

Abstract: Given a multiplicative torus A over $\bar{\mathbb{Q}}$, we study the set $A^{[m]}$ of all points of $A(\bar{\mathbb{Q}})$ satisfying a given number m of independent multiplicative equations. More generally, we study subsets of $A(\bar{\mathbb{Q}})$ given by products $\Gamma A^{[m]}$, where Γ is an arbitrary subgroup of finite rank of $A(\bar{\mathbb{Q}})$. This is related to conjectures of Zilber and Pink and follows previous work by Bombieri, Masser and Zannier.

The problem we address is the following: given an algebraic subvariety X of A , are the points of $\Gamma A^{[m]}$ Zariski-dense in X ? Of course, it depends on the way we choose m and X . For example, if $m = \dim A$, then $A^{[m]}$ is the set of all torsion points of A , so $\Gamma A^{[m]}$ is a subgroup of finite rank of $A(\bar{\mathbb{Q}})$. In this case, we can apply a theorem of Laurent to see that $X(\bar{\mathbb{Q}}) \cap \Gamma A^{[m]}$ is not Zariski-dense in X for almost all X . On the contrary, when $m \leq \dim X$ it is easy to see that the points of $A^{[m]}$ are Zariski-dense in X , no matter how we choose X .

It turns out that, for subvarieties of A of given dimension d , the optimal value of m for non-density is $m = d + 1$, *i.e.* it gives the larger $A^{[m]}$ such that $X(\bar{\mathbb{Q}}) \cap \Gamma A^{[m]}$ is not Zariski-dense in X for almost all X of dimension d . This follows from our main result, which can be stated as follows: if X is a non degenerate subvariety of A and $m = \dim X + 1$, then for any subgroup of finite rank Γ of $A(\bar{\mathbb{Q}})$, the points lying in a height-theoretic cone with non zero radius around $\Gamma A^{[m]}$ are not Zariski-dense in X . The case $\Gamma = 1$ was proved by Habegger in 2009 as a corollary to his bounded height theorem. The general case follows from sharp estimates for the Bogomolov problem alongside a new Vojta-type height inequality on $X^m(\bar{\mathbb{Q}})$. The proof of this inequality is our main achievement. It relies on diophantine approximation techniques coming from previous work by Vojta, Faltings, Bombieri, \dots , Rémond.