Mixing times of Markov chains (Math 608 E) - Jonathan Hermon

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Markov chains are random processes whose evolution depends on the past solely through their current state. An ergodic Markov chain converges to its equilibrium distribution as time goes to infinity. But how long should one wait until the distribution is "close" to the invariant one? How many times should one shuffle a deck of cards until the order becomes uniform? This question lies at the heart of the modern theory of mixing times for Markov chains.

The classical theory of Markov chains studied fixed chains and the focus was on large time asymptotics of their distribution. Recently the need to analyse large spaces has increased and the focus has shifted on studying asymptotics of the mixing time as the size of the state space tends to infinity. The area of mixing times is at the interface of mathematics, statistical physics and theoretical computer science. In statistical physics the mixing time is the time it takes the system to forget its initial state. Phase transitions often manifest as phase transitions in the mixing time. In theoretical computer science the mixing time is the main component in the running time of various randomized algorithms (e.g. for approximate counting the size of a complicated combinatorial set) and of Monte Carlo simulations.

In this course we will develop the basic theory and some of the main techniques and tools from probability (e.g. couplings), geometry (e.g. isoperimetric profile) and spectral theory used to estimate mixing times. These are indispensible techniques in the toolkit of anyone using discrete probability in their research. We will apply them to study the mixing time of several chains of interest. We shall also discuss the cutoff phenomenon which says that a Markov chain converges to equilibrium abruptly. This phenomenon seems to be widespread but it remains a challenging question to obtain criteria for cutoff for general classes of chains.

Among the key running examples will be the Ising model (a statistical physics model for ferromagnetism), and some interacting particle systems like the exclusion and the interchange processes. We shall also discuss some applications in theoretical computer science, like approximate counting.

List of topics:

- Definitions and basic properties of mixing times.

- Couplings, stopping times and strong stopping rules.
- First examples of cutoff: The hypercube and some card shuffling scemes.
- The spectral decomposition.
- Comparison techniques (canonical paths) and application (e.g. the interchange process and random walks on convex sets).
- Symmetric chains: Random walks on groups and the interchange process as a key example.
- Evolving sets and the spectral profile with applications (e.g. random walks on groups of polynomial growth).
- Cheeger's inequality.
- The Transportation Metric and Path Coupling with applications to approximate counting and for the Ising model.
- Cover times and applications to mixing time of random walks on lamplighter graphs.
- Relations between hitting times and mixing times, including a general characterization of cutoff and a neat spectral characterization for chains on trees.
- The Varopoulos-Carne theorem.
- Coupling from the past and application to the Ising model.
- Random graphs (the configuration model and random Cayley graphs).
- Sensitivity of mixing times under small perturbations.
- Cutoff for product chains.
- Wilson's method.
- Universal mixing time bounds for the exclusion process.

Pre-requisites

This course assumes almost no background, except for prior exposure to Markov chains at an elementary level. Familiarity with martingales would be useful.

Final grade: will be assigned based on assignments.

Course's website: https://www.math.ubc.ca/ jhermon (currently under construction). See also: http://www.statslab.cam.ac.uk/ jh2129/Mixing/mc.html.

When and where: Tuesday and Thursday 2-3:30 PM at Math 126.

Literature

1. D. Levin and Y. Peres with contributions by E. Wilmer, Markov chains and Mixing Times. Second edition. American Mathematical Society, 2008. Second edition available online on David Levin's website.

2. D. Aldous and J. Fill, Reversible Markov Chains and Random Walks on Graphs. Unfinished manuscript, available online at David Aldous' website.

3. R. Montenegro and P. Tetali, Mathematical aspects of mixing times in Markov chains. Foundations and Trends in Theoretical Computer Science: Vol. 1: No. 3, pp 237-354, 2006. Available online at Prasad Tetali's website.

4. Lecture notes by Perla Sousi. Available online at Perla Sousi's website.

5. Lecture notes by Nathanael Berestycki. Available at my website.