# SYLLABUS FOR 603D: HOMOTOPY THEORY AND $\mathbb{A}^1$-HOMOTOPY THEORY

## 1. Contact information

The instructor for this course is me, Ben Williams. I may be reached at `tbjw@math.ubc.ca`.

The course website is `http://www.math.ubc.ca/~tbjw/A1Homotopy/index.html`.

## 2. Time and Place

The first meeting will be in MATX 1102 on Monday 6 January 2020 at 11am. Future meetings may be at a different time, depending on the preferences of students wishing to take the course.

## 3. Description & Prerequisites

This is a topics course in topology. The intent is to assume only a reasonable amount of background, and by the end of the course to define and say something meaningful about $\mathbb{A}^1$-homotopy theory (also known as *motivic homotopy theory*. This will require skipping very many proofs in lecture, but the advantage is that the prerequisites are

- A first course in algebraic topology, including the definitions of homotopy and homotopy groups and homological algebra.
- Undergraduate algebra—rings, ideals, groups etc.
- The basics of the theory of schemes, including the Spec functor, the definition of a scheme and the theory of varieties.
- The basics of category theory, the definition of limits, colimits, adjoint functors.

If you'd like to take the course but feel you lack some of the above, let me know.

## 4. Homework & Grades

A number of homework exercises will be handed out over the course of the term. These will form the basis of the final grade for anyone registered in this course.

## 5. Outline

The course is in three parts. The provisional number of lectures for each topic is listed.

### 5.1. **Abstract and simplicial homotopy theory.**

(1) Model categories and their homotopy categories. [2]
(2) Quillen adjunctions. [1]
(3) Simplicial sets. [1]
(4) Realization of simplicial sets. [1]
(5) The model structure on simplicial sets. [2]
(6) Monoidal and simplicial model structures. [1]
(7) Homotopy colimits and limits. [2]
(8) Homotopy cofibre and fibre sequences. [1]
(9) Left Bousfield localization. [1]

5.2. **Local Homotopy Theory.**

(10) Sheaf theory for Grothendieck topologies. [2]
(11) FPPF descent. [1]
(12) The étale and Nisnevich topologies. [2]
(13) Local homotopy theory. [3]
(14) The Dold–Kan correspondence. [1]
(15) Classifying spaces and torsors. [2]

5.3. $\mathbb{A}^1$ **Homotopy Theory.**

(16) Definition of $\mathbb{A}^1$ model structures. [2]
(17) The $\mathrm{Sing}^{\mathbb{A}^1}$-construction and naïve homotopy. [2]
(18) Purity. [2]
(19) Realizations. [1]
(20) Spheres and contractions. [1]
(21) Representable cohomology theories: motivic cohomology, $K$-theory, algebraic cobordism. [3]
(22) Vector bundles from the motivic point of view. [2]

## 6. References

The following will be the principal references used during the term. This list may be updated:

- For model categories: [Hov99] and [Hir03].
- For simplicial homotopy theory, [Hov99] again, but it is largely drawn from [GJ99].
- For the theory of sheaves: [MLM92].
- For algebraic geometry, the standard reference is [Har77] but we will draw heavily on the Stacks Project, e.g. [de 16].
- For local homotopy theory: [JSS15] is a definitive source, but we will not need the full generality of this work.
- For -homotopy theory, the original source is [MV99], but there are easier-to-read introductions now, e.g. [AE16].
- For select topics in homotopy, we may look at [Caz08], [AHW17], [ST15].

## References

[AE16] Benjamin Antieau and Elden Elmanto. A primer for unstable motivic homotopy theory. *arXiv:1605.00929 [math]*, November 2016.

[AHW17] Aravind Asok, Marc Hoyois, and Matthias Wendt. Affine representability results in $\mathbb{A}^1$-homotopy theory, I: Vector bundles. *Duke Mathematical Journal*, 166(10):1923–1953, 2017.

[Caz08] Christophe Cazanave. Homotopy classes of rational functions. *Comptes rendus - Mathematique*, 346(3-4):129, February 2008.

[de 16] A. J. de Jong. Stacks Project — Chapter 7: Sites and Sheaves. http://stacks.math.columbia.edu/chapter/7, 2016. Tag 00UZ.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial Homotopy Theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Gradute Texts in Mathematics*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Hir03] Philip S Hirschhorn. *Model Categories and Their Localizations*. Number v. 99 in Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003.

[Hov99] Mark Hovey. *Model Categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[JSS15] John F. Jardine, SpringerLink (Online service), and SpringerLINK ebooks - Mathematics and Statistics. *Local Homotopy Theory*. Springer Monographs in Mathematics. Springer New York, DE, 2015.

[MLM92] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic*. Universitext. Springer-Verlag, January 1992.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Publications Mathématiques de L'Institut des Hautes Scientifiques*, 90(1):45 – 143, December 1999.

[ST15] Marco Schlichting and Girja S. Tripathi. Geometric models for higher Grothendieck–Witt groups in $\mathbb{A}^1$-homotopy theory. *Mathematische Annalen*, 362(3-4):1143–1167, August 2015.