

MATH 101 – Integral Calculus with Applications to Physical Sciences and Engineering
Learning Objectives

Course-level learning goals:

In this course students will learn the basic ideas, tools and techniques of integral calculus and will use them to solve problems from real-life applications. In particular, students will learn

- to perform integration and other operations for certain types of functions and carry out the computation fluently;
- approximation techniques for integration;
- to determine whether a sequence or a series is convergent or divergent and evaluate the limit of a convergent sequence or the sum of a convergent series;
- to recognize when and explain why such operations are possible and/or required;
- to interpret results and determine if the solutions are reasonable.

In addition, students will apply the above skills and knowledge to translate a practical problem involving some real-life applications into mathematical problem and solve it by mean of Calculus. The applications include science and engineering problems involving areas, volumes, average values, kinematics, work, hydrostatic forces, centroid, and separable differential equations. Students will also learn simple concepts involving sequences, series and power series. In general, when solving a problem students will be able to

- after reading a problem, correctly state in their own words what the problem is asking in mathematical terms and what information is given that is needed in order to solve the problem;
- after restating the problem, identify which mathematical techniques and concepts are needed to find the solution;
- apply those techniques and concepts and correctly perform the necessary algebraic steps to obtain a solution;
- interpret results within the problem context and determine if they are reasonable.

Topic-level learning goals:

Here are the major learning outcomes of the course. Not all of these outcomes are of equal importance. The level of difficulty of problems to be solved is indicated by examples given below, and also by the Suggested Homework Problems from the textbook assigned throughout the term and past exams.

1. Approximate the area between a curve and the x -axis by using the **left**, **right** or **midpoint sums**. Interpret a definite integral in terms of the area between a curve and the x -axis. Compute definite integrals by using the **Riemann sum**, the definition of a definite integral. Use the **comparison properties** to estimate the value of a definite integral. Examples:

(a) Estimate the area under the graph $y = \sqrt{x}$ from $x = 0$ to 4 using N approximating rectangles and right endpoints. Sketch your graph and the rectangles. Is your estimate an underestimate or an overestimate?

(b) Write an integral that is defined by the expression $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\pi}{n} \sin \frac{i}{4n}$.

(c) Use the definition of a definite integral to show that $\int_a^b x^2 dx = \frac{b^3 - a^3}{3}$.

(d) Show that $\frac{\sqrt{2}}{24}\pi \leq \int_{\pi/6}^{\pi/4} \cos x dx \leq \frac{\sqrt{3}}{24}\pi$.

2. Compute definite integrals by using the **fundamental theorem of calculus**. Be able to recognize functions that are given as definite integrals with variable upper and lower limits and find their derivatives, relate antiderivatives to definite and indefinite integrals, and the **net change** as the definite integral of a rate of change. Examples:

(a) Evaluate $\int_0^4 \left(\frac{x^2}{4} + \sqrt{x} + e^x \right) dx$.

(b) Differentiate $\int_{\ln x}^{e^x} \frac{1}{\sqrt{1+t^4}} dt$.

(c) A ball is thrown vertically upward from the ground at an initial velocity of 49 m/s. How high is the ball from the ground 6 seconds later? Acceleration due to gravity is 9.8 m/s^2 .

3. Compute integrals of basic functions by using antiderivative formulas and techniques such as **substitution**, **integration by parts**, **trigonometric identities**, **trigonometric substitutions**, **partial fraction decomposition** and **rationalizing substitutions**. Be able to simplify and manipulate the integrand and choose an effective technique or combination of techniques based on the form of the integrand. Examples:

(a) Evaluate $\int \frac{e^{2t}}{1 + e^{4t}} dt$.

(b) Evaluate $\int x \csc x \cot x dx$.

(c) Evaluate $\int_{\pi/6}^{\pi} \sin^2 x \cos^3 x dx$,

(d) Evaluate $\int \frac{\sqrt{4+x^2}}{x^2} dx$

(e) Evaluate $\int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6} dx$

(f) Evaluate $\int \theta \tan^2 \theta d\theta$.

(g) Evaluate $\int \frac{dx}{\sqrt{x+1} + \sqrt{x}}$.

(h) Evaluate $\int_2^6 \frac{dx}{x\sqrt{4x+1}}$.

(i) Evaluate $\int \arctan \sqrt{x} dx$.

4. Construct an integral or a sum of integrals that can be used to find the **area of a region** bound by two or more curves by considering approximating rectangles that could be **vertical** (of widths Δx) or **horizontal** (of widths Δy). Examples:

(a) Find the area of the region bounded by $y = 2 - x^2$ and $y = |x|$.

(b) Find the area of the region bounded by $y^2 = x$ and $y = 2 - x$ from $y = 2$ to $y = -2$.

5. Construct an integral or a sum of integrals that can be used to find the **volume of a solid** by considering its **cross-sectional areas**. For solids that are obtained by revolving a region about an axis of rotation, find the volume by considering **cross-sectional discs or washers**. Examples:

(a) Let R be one of the infinitely many regions bounded by $y = 1 + \sec x$ and $y = 3$. Find the volume of the solid obtained by rotating R about $y = 1$.

(b) Find the volume of the solid by rotating the region bounded by $y = x^2$ and $y = x + 2$ about the line $x = 3$.

(c) Consider a cone with base radius of r cm and height of h cm. Use the method of cross sections to show that the volume of the cone is $\frac{1}{3}\pi r^2 h$ cm³.

(d) The base of a solid S is the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Each cross-section of S perpendicular to the y -axis is a right isosceles triangle with hypotenuse on the xy -plane. Find the volume of S .

6. Construct an integral or a sum of integrals that can be used to find the total **work done in moving an object** either by considering the entire object moving over an infinitesimal distance or by considering an infinitesimal section of the object moving over the entire distance. Examples:

(a) If 6 J of work is needed to stretch a spring from 10 cm to 12 cm and another 10 J is needed to stretch it from 12 cm to 14 cm, what is the natural length of the spring?

(b) A 10-kg bucket containing 36 kg of water is lifted from the ground to a height of 12 m with a rope that weigh 0.8 kg/m. How much work is done?

(c) A spherical tank of radius 5 m is built underground with the top 3 m below ground level. If the tank is full of water, how much work is needed to pump all the water out to the ground level? Assume that the density of water is 1000 kg/m³.

7. Compute the **average value of a function** on an interval. Examples:

(a) Find the average value of $f(x) = \sin x$ on $[0, \pi/2]$.

- (b) For what value of a does the average value of $f(x) = \frac{1}{(3+2x)^2}$ on $[-1, a]$ equal to $\frac{1}{50}$.
- (c) If a freely falling body starts from rest, then its displacement is given by $s = \frac{1}{2}gt^2$. Let the velocity after a time T be v_T . Show that if we compute the average of the velocities with respect to t we get $v_{\text{avg}} = \frac{1}{2}v_T$, but if we compute the average of the velocities with respect to s we get $v_{\text{avg}} = \frac{2}{3}v_T$.
8. Approximate the value of a definite integral using the **midpoint rule**, the **trapezoidal rule**, and **Simpson's rule** to a desired accuracy. Examples:
- (a) Use the trapezoidal rule to approximate $\int_4^6 \ln(x^3 + 2) dx$ with $n = 10$. Estimate the error of the approximation.
- (b) How large should n be to guarantee that the Simpson's rule approximation to $\int_0^1 e^{x^2} dx$ is accurate to within 1×10^{-5} ?
9. Determine whether an **improper integral** (which either has infinite lower or upper limits of integration, or has a integrand with infinite discontinuities within or at the boundary of the interval of integration) diverges or converges, by evaluating the improper integral or by using the comparison theorem. Examples:
- (a) Determine whether $\int_0^\infty \frac{dx}{x^2 + 3x + 2}$ is divergent or convergent. If it is convergent, evaluate the integral.
- (b) Find the area of the region $\{(x, y) | 0 \leq x \leq \pi/2, 0 \leq y \leq \tan x\}$.
- (c) Determine whether $\int_{-\infty}^\infty \frac{\sin x}{1 + x^2} dx$ is convergent or divergent.
10. Construct an integral or a sum of integrals that can be used to find the **moment** about the x - or y - axis and the x - or y -coordinate of the **centroid** of a plane region. Examples:
- (a) Find the moments and the centre of mass of a plate with shape $\{(x, y) | x \geq 0, y \geq x - 1, x^2 + y^2 \leq 1\}$ if the density of the plate is 2.
- (b) Let E be the ellipse $x^2 + k^2y^2 = 1$, where k is a constant and $k > 1$. Let S be the region inside the circle $x^2 + y^2 = 1$, outside E , and above the x -axis. Find all values of k such that the centre of mass of S lies inside S .
11. Construct a **first-order differential equation** with appropriate **initial condition** that can be used to model a quantity described in a problem by finding a relationship between the quantity and its derivative. Find a function that solves a **separable differential equation** with or without an initial condition. Examples:

- (a) Solve the initial-value problem $\frac{dy}{dx} = y^4(x+1)^2$, $y(0) = -1$. Express your answer in the form $y = f(x)$ and simplify.
- (b) The population of fish in a lake is m million, where $m = m(t)$ varies with t (in years). The number of fish is currently 2 million. If m satisfies $\frac{dm}{dt} = 16m\left(1 - \frac{m}{4}\right)$, when will the number of fish equal 3 million?
- (c) A tank contains 1000 L of pure water. Brine that contains 0.05 kg of salt per litre of water enters the tank at a rate of 5 L/min. Brine that contains 0.04 kg of salt per litre of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at a rate of 15 L/min. How much salt is in the tank after 1 hour?
12. Conjecture a formula or a recurrence relation which defines all the terms of a **sequence**. Determine if a sequence is **convergent**, **divergent**, **increasing**, **decreasing** or **bounded**. Evaluate the **limit** of a convergent sequence. Examples:

(a) Does the sequence $\left\{ \frac{(-1)^n n^3}{4n^3 + 3n^2 + 2n + 1} \right\}_{n=0}^{\infty}$ converge? If so, what is the limit?

(b) Is the sequence $a_n = \frac{n}{n^2 + 1}$ increasing, decreasing or non-monotonic? Is the sequence bounded?

13. Be able to use the **sigma notation** to represent a **series**, and recognize that the sum of the series is the limit of the partial sums. Find the sum of a **geometric series** or determine that a geometric series is divergent. Be able to recognize **telescoping sums** and **harmonic series**. Use the test for divergence to determine if a series can be concluded to be divergent. Examples:

(a) Evaluate $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$ or determine that the sum is divergent.

(b) Evaluate $\sum_{n=1}^{\infty} \arctan n$ or determine that the sum is divergent.

(c) Evaluate $\sum_{n=0}^{\infty} \frac{e^n}{3^{n+1}}$ or determine that the sum is divergent.

(d) Evaluate $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ or determine that the sum is divergent.

14. Use the **integral test** to determine if a series is convergent or divergent. Use integrals to approximate the sum of a series to a desired accuracy. Examples:

(a) Determine whether the series $\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \dots$ is convergent or divergent.

(b) Find all the values of p for which $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is convergent.

15. Use the **comparison test** to determine if a series is convergent or divergent by proposing a different series whose convergence properties can be easily determined.

(a) Determine whether the series $\sum_{n=1}^{\infty} \frac{3}{n + \sqrt{n}}$ is convergent or divergent.

(b) Determine whether the series $\sum_{n=1}^{\infty} \frac{n + 2^n}{n^2 2^n}$ is convergent or divergent.

16. Be able to recognize **alternating series**. Use the alternating series test to determine whether a series can be concluded to be convergent. Use the **alternating series estimation theorem** to approximate the sum of an alternating series. Examples:

(a) Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ is convergent or divergent.

(b) How many terms of the series $\sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{5^n}$ do we need to add in order to find the sum with error within 0.0001?

(c) Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n = \frac{1}{n}$ if n is odd and $b_n = \frac{1}{n^2}$ if n is even, is divergent. Why does the alternating series test not apply?

17. Be able to distinguish between **absolute convergence** and **conditional convergence** of a series. Use the **ratio test** to determine whether a series is absolutely convergent or divergent. Examples:

(a) Determine whether $\sum_{n=0}^{\infty} (n + 1)4^{2n+1}$ is absolutely convergent, conditionally convergent, or divergent.

(b) For what values of p does $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge absolutely? conditionally?

(c) Show that the ratio test is inconclusive for both $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1 + n^2}$ and $\sum_{n=1}^{\infty} \frac{n}{1 + n^2}$. Are these series convergent or divergent?

18. Be able to recognize functions that are represented as a **power series**. Determine the **radius of convergence** and the **interval of convergence** of a power series. Use the formula for **Taylor** (or **Maclaurin**) **series** to find a power series representation of a function. Examples:

(a) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n (x - 3)^n}{2n + 1}$.

(b) Show that if $\lim_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} = c \neq 0$, then the radius of convergence of the power series

$$\sum_{n=0}^{\infty} c_n x^{2n} \text{ is } R = \frac{1}{\sqrt{c}}.$$

- (c) Find the Taylor series for $f(x) = \sqrt{x}$ at $x = 9$ and determine its radius of convergence.
19. Find the power series representation of a function by manipulating, **differentiating**, **integrating** other power series. Use partial sums or the **remainder** of Taylor series to approximate a value of a function or a definite integral to a desired accuracy. Prove that a function is equal to its power series expansion. Use power series to evaluate limits. Examples:

(a) Find the power series representation for $f(x) = \frac{x^3}{(x-2)^2}$ by using the power series for $\frac{1}{1-x}$ and determine the radius of convergence.

(b) Find the Maclaurin series for $f(x) = x \cos\left(\frac{1}{2}x^2\right)$.

(c) Use power series to approximate $\int_0^{0.3} \frac{x^2}{1+x^4} dx$ with error within 1×10^{-6} .

(d) Prove that for $0 < x < 4$, $f(x) = \frac{1}{x}$ is equal to its Taylor series expansion at $x = 2$.

(e) Evaluate $\lim_{x \rightarrow 0} \frac{x - \arctan x}{x^3}$.