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The University of British Columbia
Final Examinations - December 2006

Mathematics 221: *Matrix Algebra*

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Closed book examination. **Time: 2.5 hours = 150 minutes.**

Special Instructions: **No aids allowed.** Write your answers in the answer booklet(s). If you use more than one booklet, put your name and the number of booklets used on each booklet. **Show enough of your work to justify your answers.**

1. (8 points) Find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix}$.

2. (7 points) Find the determinant of the matrix $\begin{bmatrix} 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{bmatrix}$.

3. (15 points) Consider the matrix $A = \begin{bmatrix} 1 & 7 & 5 \\ 2 & 15 & h \\ 0 & 0 & 0 \end{bmatrix}$, where h is an unspecified number.

(a) Find a vector in the column space of A and a vector in the null space of A.

(b) Is the vector $\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ in the column space of A? Why or why not?

(c) Find the rank of A and the dimension of $\mathrm{Nul}(A)$.

(d) Are the first two column vectors in A orthogonal to each other? Find the length of the first column vector in A.

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4. (15 points) Consider the system of equations

$$\begin{array}{ccccccccc} x & & & + & z & & & = & p \\ & & y & & & + & 2w & = & 0 \\ x & & & + & 2z & + & 3w & = & 0, \\ & & 2y & + & 3z & + & qw & = & -6 \end{array}$$

where the constants p and q are *not* specified. For which values of p and q, if any, does this system have:
(i) No solution?
(ii) Exactly one solution?
(iii) Exactly two solutions?
(iv) More than two solutions?
Remember to provide some calculations and/or other reasons to support your answers.

5. (15 points) Consider two linear transformations, one that rotates each vector in $\mathbb{R}^2$ by $+45°$, and one that projects each vector in $\mathbb{R}^2$ into the x_1-axis. The standard matrices S for that rotation and P for that projection are

$$S = \begin{bmatrix} 1/\sqrt{2}, & -1/\sqrt{2} \\ 1/\sqrt{2}, & 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

(a) Find the standard matrix, V say, for the linear transformation on $\mathbb{R}^2$ that firsts rotates each vector by $+45°$, and then projects the result of that step into the x_1-axis
(b) Find the standard matrix, T say, for the transformation on $\mathbb{R}^2$ that does the two steps above, and then rotates the resulting vector by $-45°$.
(c) Is the transformation mapping each $\vec{x}$ in $\mathbb{R}^2$ to $T\vec{x}$ one-to-one? Does that transformation map $\mathbb{R}^2$ onto $\mathbb{R}^2$? Explain your answers briefly.

6. (15 points) Let f_k and g_k denote the populations of foxes and geese in a park in year k. Suppose that in the following year those populations f_{k+1} and g_{k+1} are given by

$$\begin{bmatrix} f_{k+1} \\ g_{k+1} \end{bmatrix} = C \begin{bmatrix} f_k \\ g_k \end{bmatrix}, \quad \text{where} \quad C = \begin{bmatrix} 0.2 & 0.2 \\ -p & 1.3 \end{bmatrix}.$$

Here p a number determined by the rate at which the foxes catch the geese.
(a) In this part and the next one, let $p = 0.9$. Confirm that $\begin{bmatrix} 2 \\ 9 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are eigenvectors of the matrix C, and find the corresponding eigenvalues.
(b) Suppose that the initial populations are $f_0 = 4$ and $g_0 = 11$. Suppose again that $p = 0.9$. What happens *in this model* to the populations as $k \to \infty$?
(c) Now let $p = 1.2$, and again let $f_0 = 4$ and $g_0 = 11$. In this model, what happens to the populations as $k \to \infty$?

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7. (15 points) Assume that the matrix $H = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ has eigenvalues 0 and 3.

(a) Find all eigenvectors for each of those eigenvalues.

(b) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of H. Is the matrix H diagonalizable?

(c) Give a reason why there is an orthogonal set of eigenvectors of H that form a basis for $\mathbb{R}^3$, or give a reason why there is no such orthogonal basis.

8. (10 points) Give brief explanations for the following facts.

(a) If a matrix X has an inverse, but a matrix Y does *not* have an inverse, then any matrix Z satisfying the equation $ZX = Y$ has no inverse either.

(b) If a matrix B has an inverse, then the determinant of B^{-1} can *not* be equal to 0.

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