MATH 422/501
FINAL EXAM

Name:
Student ID:

Exam rules:

- You can refer to any result that was proved in class or that appeared in a homework. Ask if you want to refer to some other result.
- There are 16 problems in this exam. Each problem is worth 5 marks.

PROBLEM 1. Let G be a finite abelian group that is not cyclic. Prove that G contains a subgroup isomorphic to $\mathbb{I}_p \oplus \mathbb{I}_p$ for some prime p.

Problem 2. Let P be a normal Sylow p-subgroups of a finite group G. Prove that if $\phi : G \to G$ is a group homomorphism, then $\phi(P) \subset P$.

PROBLEM 3. Prove that a group of order $31 \cdot 32$ cannot be simple.

PROBLEM 4. Let G be a finite group, such that the group of automorphisms $Aut(G)$ is cyclic. Prove that G is abelian. (Hint: There is a homomorphism $G \to Aut(G)$. Study the kernel and the image of this homomorphism.)

PROBLEM 5. Prove that there is no simple group G of order 300. (Hint: Let G act on its Sylow 5-subgroups.)

Problem 6. Let G be a p-group, $|G| = p^n$ for some prime p and integer n. Let N be a normal subgroup of G of order p. Prove that N lies in the center of G. (Hint: Let G act on N by conjugation and consider the orbits of this action.)

PROBLEM 7. Let I be an ideal in a ring R (commutative, with 1) and define

$$Rad(I) = \{r \in R | r^n \in I \text{ for some } n > 0\}.$$

1. Prove that $Rad(I)$ is an ideal.

2. If $R = \mathbb{Q}[x]$ and $I = (f(x))$, describe a generator of $Rad(I)$ in terms of irreducible factors of $f(x)$.

PROBLEM 8. Let F be a field and $f(x), g(x) \in F[x]$ irreducible polynomials of degree 6 and 7, respectively. Let α be a root of $f(x)$ in some extension field. Prove that $g(x)$ is irreducible in $F(\alpha)[x]$.

PROBLEM 9. Let $f(x) = (x^3 - 2)(x^2 - 5) \in \mathbb{Q}[x]$. Let E be the splitting field of $f(x)$ over $\mathbb{Q}$. Find an intermediate field $\mathbb{Q} \subset K \subset E$, such that $Gal(E/K) = \mathbb{I}_6$.

PROBLEM 10. Let $p \neq 2$ be a prime and let $\xi \in \mathbb{C}$ be a primitive p-th root of 1.

1. Show that

$$[\mathbb{Q}(\xi) : \mathbb{Q}(\xi + \xi^{-1})] = 2.$$

2. Show that $\mathbb{Q} \subset \mathbb{Q}(\xi + \xi^{-1})$ is a Galois extension and find its Galois group.

Problem 11. Let F be a finite field, $f(x) \in F[x]$ an irreducible polynomial, and E the splitting field of $f(x)$. If $\alpha \in E$ is a root of $f(x)$, prove that $E = F(\alpha)$. (Hint: Is $F(\alpha)$ Galois over F?)

PROBLEM 12. We proved in class that the primitive element theorem holds for finite extensions of a finite field $\mathbb{F}_q$.

1. Prove that $\mathbb{F}_q[x]$ contains irreducible monic polynomials of every degree $n > 0$.

2. Prove that $\mathbb{Q}[x]$ contains irreducible monic polynomials of every degree $n > 0$.

PROBLEM 13. Prove that $\mathbb{F}_{p^n}$ contains a primitive m-th root of 1 if and only if m divides $p^n - 1$.

PROBLEM 14. Prove that if a Galois extension $F \subset E$ has Galois group S_6, then there is no intermediate field $F \subset K \subset E$, such that $[K : F] = 5$.

PROBLEM 15. Let F be a field, $f(x) \in F[x]$ an irreducible polynomial of degree 4, such that the Galois group of $f(x)$ is S_4. If α is a root of $f(x)$ in a splitting field, prove that there is no intermediate field

$$F \subsetneq K \subsetneq F(\alpha).$$

Problem 16. Let p be a prime, and let F be a field of characteristic 0, such that every irreducible polynomial $g(x) \in F[x]$ has degree p^n for some n. Prove that every polynomial $f(x) \in F[x]$ is solvable by radicals.