Problem 1.
Define the following terms:
(a) centre of a group
(b) normal subgroup
(c) solvable group
(d) Sylow subgroup
(e) minimal polynomial
(f) Galois extension
(g) separable polynomial

Problem 2.
Carefully state:
(a) the orbit equation,
(b) a result describing the orbits of an action of the Galois group of the splitting field of a polynomial,
(c) a criterion by which to recognize Galois extensions.

Problem 3.
Give examples of the following phenomena (without proofs):
(a) a non-trivial normal subgroup of the group of isometries of the Euclidean plane,
(b) a Sylow 5-subgroup which is not normal,
(c) a finite group which is not solvable,
(d) a field with 9 elements,
(e) a field extension of degree 3 which is not Galois,
(f) a field extension of degree 2 which is not Galois,
(g) a Galois extension with cyclic Galois group of order 3.
Problem 4.
Let \((\mathbb{Z}/n\mathbb{Z})^*)\) be the multiplicative group of the ring \(\mathbb{Z}/n\mathbb{Z}\). In other words, \((\mathbb{Z}/n\mathbb{Z})^*)\) is the group of residue classes \(a + n\mathbb{Z}\) modulo \(n\), such that \(a\) is coprime to \(n\), with multiplication as group operation. Let \(G\) be a cyclic group of order \(n\). Prove that \(\text{Aut}(G)\) is isomorphic to \((\mathbb{Z}/n\mathbb{Z})^*)\).

Problem 5.
Let \(G\) be the group of rotational symmetries of the cube.
(a) Prove that \(G\) contains 24 elements.
(b) How many Sylow 3-subgroups does \(G\) contain? Describe them geometrically.
(c) How many Sylow 2-subgroups does \(G\) contain? Describe them geometrically.

Problem 6.
Prove that the fields \(\mathbb{Q}(\sqrt{2}) \subset \mathbb{R}\) and \(\mathbb{Q}(\sqrt{3}) \subset \mathbb{R}\) are not isomorphic.

Problem 7.
Let \(K\) be a field and assume that the characteristic of \(K\) does not divide the integer \(n > 0\). Let \(L/K\) be a splitting field of the polynomial \(x^n - 1 \in K[x]\). Denote by \(\mu_n \subset L\) the group of \(n\)-th roots of unity in \(L\).
(a) Prove that \(L/K\) is a Galois extension. Let \(G\) be the Galois group.
(b) Construct an injective homomorphism of groups \(G \rightarrow \text{Aut}(\mu_n)\).
(c) Conclude that \(G\) is abelian.

Problem 8.
Let \(\zeta = e^{2\pi i/12}\) and \(K = \mathbb{Q}(\zeta) \subset \mathbb{C}\).
(a) Prove that \(K/\mathbb{Q}\) is a Galois extension. Let \(G\) be the Galois group.
(b) Prove that \(G\) is isomorphic to the Kleinian 4-group.
(c) Find the minimal polynomial of \(\zeta\) over \(\mathbb{Q}\).
(d) Factor the polynomial \(x^{12} - 1 \in \mathbb{Q}[x]\) into irreducible factors.