Be sure this exam has 7 pages including the cover

The University of British Columbia

Sessional Exams – 2007/2008 Winter Term 1
Mathematics 420/507 Real Analysis I
Measure theory and Integration

Name: ______________________________

Student Number: ______________________________

This exam consists of **6** questions worth **100** marks in total. No aids are permitted.

Problem	max score	score
1.	25	
2.	15	
3.	15	
4.	15	
5.	10	
6.	20	
total	100	

Do not define terms in theorems unless explicitly requested.
If short of time, show good judgment by focusing on key steps.

(25 points) 1. (a) Define: the Cartesian product $X = \prod_{\alpha \in \mathcal{A}} E$. What does the axiom of choice say about X?

(b) Define: σ algebra; Borel measure on $\mathbb{R}^n$.

(c) State: the Carathéodory extension theorem for an outer measure, defining terms in it that are not already defined.

(15 points) 2. (a) Define: *regular* as in regular Borel measure.

(b) Let $\mathcal{A}$ be an algebra, μ be a finite measure on $\sigma(\mathcal{A})$, $B \in \sigma(\mathcal{A})$, and $\epsilon > 0$. Prove there exists $A \in \mathcal{A}$ with $\mu(A \Delta B) < \epsilon$.

(15 points) 3. (a) State the Fatou and Dominated Convergence Lemmas.

(b) Prove that $\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{n}{1+(nx)^2}\sin(x)\,dx$ exists and evaluate it. (First think about the graph of $\frac{n}{1+(nx)^2}$).

(c) Let f be continuously differentiable. Prove that $\lim_{n\to\infty}\int_0^1 n\big(f(x+1/n)-f(x)\big)\,dx$ exists and evaluate it. The mean value theorem may be useful.

(15 points) 4. (a) State the Lebesgue-Radon-Nikodym theorem.

(b) Let $(X, \mathcal{M}, \mu)$ be a finite measure space, let $\mathcal{N}$ be a sub-σ-algebra of $\mathcal{M}$, and let ν be the restriction of μ to $\mathcal{N}$. If $f \in L^1(\mu)$ prove there exists $g \in L^1(\nu)$ such that $\int_E f\,d\mu = \int_E g\,d\nu,\ \forall E \in \mathcal{N}$.

(c) Give a counter-example to the conclusion of the previous part when μ is not finite.

(10 points) 5. (a) State the Lebesgue differentiation theorem.

(b) Let $f \in L^1$ and equal to zero in an open interval containing 0. Prove that $\lim_{n\to\infty} \int_0^y n\big(f(x+1/n) - f(x)\big)\,dx$ exists for almost all y and evaluate it. Hint. $\int_0^y nf(x+1/n)\,dx - \int_0^y nf(x)\,dx$.

(20 points) 6. (a) Define the L^∞ norm of a measurable function f defined on a measure space $(X, \mathcal{M}, \mu)$.

(b) Let $E_a = \{x : |f(x)| \leq a\}$. For $a = \|f\|_\infty$ prove that $\mu\big(E_a^c\big) = 0$, and for all $b < a$, $\mu\big(E_b^c\big) > 0$.

(c) Prove that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ for $f, g \in L^\infty$.

(d) Let X be an uncountable set. Let $\mathcal{M}$ be the sigma algebra of sets E such that either E is countable or E^c is countable. Let μ be counting measure. Prove that if $f \in L^\infty$ then (a) f is bounded by $\|f\|_\infty$ and (b) there exists a countable set E such that f is constant on E^c.