MATH 418/544, Final exam Due 2013-12-17, 10:00

Problem 1. a. Suppose X is uniform in $[0, 1]$. Find functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ so that $f_i(X)$ are both uniform in $[0, 1]$ and independent.
b. Repeat with an infinite sequence of functions f_i so that $f_i(X)$ are all independent.

Problem 2. For two random variables X, Y, prove that

$$\operatorname{Var}(XY) \leq 2\operatorname{Var}(X)\|Y\|_\infty^2 + 2\operatorname{Var}(Y)\|X\|_\infty^2.$$

(Recall $\|X\|_\infty$ is the minimal a so that $\mathbb{P}(|X| \leq a) = 1$).)

Problem 3. For i.i.d. X_n which are symmetric and not constant (X and $-X$ have the same distribution), let $S_n = \sum_{i \leq n} X_i$. Prove that a.s. $\limsup S_n = \infty$, and $\liminf S_n = -\infty$.

Problem 4. Let $f(t) = \mathbb{E}e^{tX}$.
a. Give an example of a random variable where $f(t) = \infty$ for every $t \neq 0$.
b. Show that for any random variable X the set $\{t \in \mathbb{R} : f(t) < \infty)\}$ is an interval (possibly all of $\mathbb{R}$), and that f is infinitely differentiable in the interior.

Problem 5. Find (with proof) all possible joint distributions for independent random variables X, Y that are rotationally symmetric, i.e. if $A = \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}$ is a rotation matrix then the vector $(X, Y)A$ has the same distribution as (X, Y).

Partial marks will be given for proofs with some assumptions on the variables.

Problem 6. A fair coin is tossed repeatedly. Find the expected time until the sequence HTTHTTHTTH appears as follows:
a. Consider the sequence $M_n = n - \sum_{i \in A_n} 2^i$, where $i \in A_n$ if the first i letters of the sequence are the results of coins $n - i + 1, \dots, n$. Show that M_n is a martingale.
b. Use this to find the required time. Justify all steps.
c. Write your answers for a general pattern if ypossible.

Problem 7. Consider a simple random walk on $\mathbb{Z}$. We have shown that it is recurrent. Let T be the time it takes to return to 0.
a. Show $\mathbb{E}T = \infty$.
b. What can you say about $\mathbb{P}(T > n)$ for large n?
c. Extend your answers to a random walk where the steps have $\mathbb{P}(X = 2) = \frac{3}{5}$, $\mathbb{P}(X = -3) = \frac{2}{5}$.

***Problem 8.** Suppose X_i are independent with finite expectation. Suppose $Y = \sum X_i$ converges a.s., and that $\sum \mathbb{E}X_i$ also converges. Is it necessarily the case that $\mathbb{E}Y = \sum \mathbb{E}X_i$?

Partial credit for proving the identity under more assumptions (depending on the assumptions).