**The University of British Columbia**

Final Examination - December, 2011

**Mathematics 405/607E**

Instructor: Dr. Lisa Gordeliy

Closed book examination | Time: 2.5 hours

**Last Name:** ________________ , **First:** ____________ **Signature** ________________

**Student Number** ________________

**Special Instructions:**

- Be sure that this examination has 11 pages. Write your name on top of each page.
- A formula sheet is provided. No calculators or notes are permitted.
- Show all your work and make your reasoning clear.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

**Rules governing examinations**



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|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| Total | | 50 |

**Useful formulas**

- The **Taylor series** expansion of $f(x)$ about $x_0$ to $n+1$ terms is given by

$$f(x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

  in which $x_0 \leq \xi \leq x$.

- The **Newton's algorithm** approximates the solution of a system of $n$ nonlinear equations in $n$ unknowns $u_1$, ..., $u_n$:

$$R(U) = \begin{bmatrix} R_1(U) \\ \vdots \\ R_n(U) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{where } U = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix},$$

  via iterations,

$$U^{(k)} = U^{(k-1)} - \left[J\left(U^{(k-1)}\right)\right]^{-1} R\left(U^{(k-1)}\right)$$

  in which the Jacobian matrix is defined from

$$J(U) = \frac{\partial R}{\partial U}, \quad J(U)_{ij} = \frac{\partial R_i}{\partial u_j}.$$

**Problem 1 (10 points)**

Consider a natural cubic spline $s(x)$ on $[0, 2]$ defined by

$$s(x) = \begin{cases} 1 + 2x - x^3 & 0 \le x \le 1, \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 & 1 \le x \le 2. \end{cases} \tag{1}$$

(a) Find $a$, $b$, $c$ and $d$.

(b) Is there a function $f(x)$ for which (1) represents the clamped spline interpolating at points $x = 0,\ 1,\ 2$? If no, explain why. If yes, state what conditions $f(x)$ must satisfy.

**Problem 2 (10 points)**

(a) By expanding $f(x)$ about $x_0$ in a Taylor series, derive the following expression for the error involved in the midpoint approximation:

$$\int_{x_0-h/2}^{x_0+h/2} f(x)dx = h\,f\,(x_0) + \frac{f^{(2)}(\xi)}{24}h^3, \quad \xi \in [x_0 - h/2, x_0 + h/2] \tag{2}$$

(b) Using the above expression (2) for the error, explain why the composite midpoint rule applied directly will not yield a very good approximation to the following integral:

$$\int_0^\pi \frac{\cos(x)}{x^{1/3}}dx \tag{3}$$

Explain what needs to be done to obtain a better approximation with the composite midpoint rule to the integral in (3). Support your explanation with derivations.

(c) The Gauss-Legendre quadrature formula with $m = 2$ integration points evaluates integrals of the form:

$$\int_{-1}^{1} f(x)dx$$

exactly if $f(x)$ is a polynomial of degree $2m - 1 = 3$. Use this fact to determine the integration points $x_i$ and the weights $w_i$ for the integration formula

$$\int_{-1}^{1} f(x)dx \approx w_1 f(x_1) + w_2 f(x_2)$$

By symmetry you may assume in your derivation that $w_1 = w_2$ and that $x_1 = -x_2$.

**Extra page**

**Problem 3 (10 points)**

Consider the Forward Euler method

$$Y_{n+1} = Y_n + h\, f\,(t_n, Y_n) \tag{4}$$

for solving the initial value problem $y'(t) = f(t, y),\ y(0) = y_0$.

(a) Derive an expression for the truncation error $T_n = \frac{y_{n+1}-y_n}{h} - f\,(t_n, y_n)$ of this method, where $y_n$ is the exact solution at $t = t_n$. What order of the truncation error do you get?

(b) By considering the model problem $f(t, y) = \lambda y$, determine the stability region of this method. Present your derivations.

(c) What is the largest value of $h$ that you can use for the problem with $f(t, y) = (-1 + \mathrm{i})y$, where $\mathrm{i} = \sqrt{-1}$, before the algorithm (4) becomes unstable? Justify your answer.

(d) What is the largest value of $h$ that you can use and maintain stability with the algorithm (4) to solve the problem with $f(t, y) = (\cos(2t) - 5)y - 2t$? Justify your answer.

(e) What is the largest value of $h$ that you can use and maintain stability with the Forward Euler method to solve the following system of ordinary differential equations? Justify your answer.

$$\begin{aligned} x'(t) &= -2x(t) + 3y(t) \\ y'(t) &= -y(t) \\ x(0) &= x_0, \quad y(0) = y_0 \end{aligned}$$

**Extra page**

**Problem 4 (10 points)**

Consider the boundary value problem for $u(x)$ with $x \in (0, 1)$:

$$u'' + u' - u^2 = 0, \qquad u(0) = 1, \quad u(1) = 2. \tag{5}$$

(a) Discretize (5) with the second-order finite difference method using a uniform mesh of nodes $x_n = hn$, where $n = 0, ..., N$, and $h = 1/N$. Write down the resulting system of nonlinear equations in the matrix form. Be explicit about the size of the matrix and how the boundary conditions are taken into account.

(b) Explain how the Newton's algorithm can be used to solve the system of nonlinear equations that you obtained in (a). Present an explicit expression for the residual and derive the Jacobian involved in the Newton's algorithm.

**Extra page**

**Problem 5 (10 points)**

Consider the one dimensional wave equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \tag{6}$$

(a) Use central differences in space on a uniform mesh $x_n = nh$, with $n = 0, \ldots, N$, to arrive at a system of ODE (ordinary differential equations):

$$\frac{du_n(t)}{dt} = A\, u_n \qquad n = 1, \ldots, N$$

for $u_n(t) \approx u(x_n, t)$, that approximates (6) via an operator $A$.

(b) Given that $\left\{e^{\mathrm{i}\xi x_n}\right\}$ are the eigenfunctions of the operator $A$, where $-\pi \leq \xi h \leq \pi$ and $\mathrm{i} = \sqrt{-1}$, derive the eigenvalues of $A$. (The eigenvalues will be purely imaginary numbers.)

(c) Will the Forward Euler method be a stable algorithm to solve this system of ODE? Will the Backward Euler method be a stable algorithm to solve this system of ODE? Justify your answer.

(d) Given that the stability region for the Leapfrog method is $\{z : z = \mathrm{i}v,\ |v| \leq 1\}$, what is the maximum timestep that can be used to solve the above system of ODE using this method? Justify your answer.

**Extra page**