MATH 401 FINAL EXAM – April 23, 2010

No notes or calculators allowed. **Time: 2.5 hours.** **Total: 100 pts.**

1. Consider the problem

$$\begin{cases} u'' + u = f(x), \quad 0 < x < \pi/2 \\ u(0) = 0, \quad u(\pi/2) = 0 \end{cases} \tag{1}$$

   (a) (4 pts.) Write down the problem that the Green's function $G(x; y)$ for problem (1) should solve.

   (b) (7 pts.) Find the Green's function $G(x; y)$ for problem (1), and express the solution $u(x)$ in terms of it.

   (c) (6 pts.) Find the solvability condition on f if the boundary conditions are changed to $u(0) + u'(\pi/2) = 0, \quad u(\pi/2) = 0$.

2. Let D be a bounded (and smooth, and open) region in $\mathbb{R}^n$, and consider the following Poisson boundary-value problem:

$$\begin{cases} \Delta u = f(x) & \text{in } D \\ u = g(x) & \text{on } \partial D \end{cases} \tag{2}$$

(for given smooth functions f and g on D and ∂D respectively).

(a) (6 pts.) Write down the problem that the Green's function $G(x; y)$ for problem (2) should solve, and express the solution $u(x)$ in terms of G.

(b) (6 pts.) Derive an expression for the Green's function $G(x; y)$ in terms of an orthonormal family of eigenfunctions $\phi_j(x)$, $j = 1, 2, 3, \ldots$, of Δ on D (with zero boundary conditions), and the corresponding eigenvalues λ_j.

(c) (5 pts.) Suppose that $f(x) \equiv 0$, and $g(x) \geq 0$ with $g(x_0) > 0$ for some $x_0 \in \partial D$. Show that the solution u of (2) satisfies $u(x) > 0$ for $x \in D$. (Hint: maximum principle).

3. (17 pts.) Consider the following problem for the wave equation on the half-line with a Neumann boundary condition:

$$\begin{cases} u_{tt} = u_{xx} & x > 0, \quad t > 0 \\ u_x(0,t) = 0, & u \to 0 \text{ as } x \to +\infty \\ u(x,0) = u_0(x), & u_t(x,0) = 0 \end{cases}$$

where $u_0(x)$ is a smooth function tending to 0 as $x \to +\infty$. Find the Green's function for this problem, and use it find the solution $u(x,t)$. (Hint: recall the Green's function for the wave equation on the entire line is $G_{\mathbb{R}}(x,t;y,\tau) = \frac{1}{2}H(t - \tau - |y - x|)$ where H is the Heavyside step function.)

4. (a) (6 pts.) Derive (from first principles) the Euler-Lagrange equation, as well as the boundary conditions, satisfied by solutions of this variational problem:

$$\min_{u\in C^2([0,1])} \int_0^1 F(u(x), u'(x), x)dx.$$

(b) (6 pts.) Find the minimizing function for the problem

$$\min_{u(0)=0,\ u(1)=1} \int_0^1 \left([u'(x)]^2 + [u(x)]^2\right) dx$$

(c) (5 pts.) Write a variational problem for functions $u(x)$, $x \in [0,1]$ whose Euler-Lagrange equation is $u'' = -\sin(u)$. (Don't worry about boundary conditions.) (Remark: this is the "nonlinear pendulum equation" with x as time, and u as angle of displacement.)

5. Let D be a bounded domain in $\mathbb{R}^n$, $p(x) > 0$ a smooth function on D, and consider the (Dirichlet) eigenvalue problem

$$\begin{cases} -\nabla \cdot [p(x)\nabla\phi] = \lambda\phi & \text{in } D \\ \phi = 0 & \text{on } \partial D \end{cases} . \tag{3}$$

(a) (3 pts.) Explain how to get an upper bound for the first eigenvalue λ_1 of problem (3) using a smooth trial function $v(x)$ on D with $v = 0$ on ∂D.

(b) Now suppose $D = D_1$ is the unit disk in $\mathbb{R}^2$, $D_1 = \{\, (x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$, and suppose $p(x) = 1 + |x|^2$.

i. (5 pts.) Find an upper bound on λ_1 using trial function $v(x) = 1 - |x|^2$. (Hint: do the integrals in polar coordinates).

ii. (5 pts.) Find a lower bound on λ_1 by comparing D_1 with an appropriate square.

iii. (4 pts.) *Explain* how you might go about getting an upper bound for the *second* eigenvalue λ_2 by using two trial functions (for example, $v(x) = 1-|x|^2$ and $w(x) = 1 - |x|^4$) – but do not try to do any computations.

6. (15 pts.) Let D_1 be the unit disk in $\mathbb{R}^2$. Use a Rayleigh-Ritz-type approach to find an approximate solution to the variational problem

$$\min_{u \in C^2(D_1),\, u(x) \equiv 1 \text{ on } \partial D_1} \int_{D_1} \left(|\nabla u|^2 + e^u \right) dx$$

by considering the one-parameter family of trial functions

$$u(x) = 1 + a(1 - |x|^2), \qquad a \in \mathbb{R}.$$

Reduce the problem to an algebaric equation for a, but do not try to solve this equation. (Hint: again, polar coordinates might help).