Be sure that this examination has three pages.

The University of British Columbia
Final Examinations – April 2007

Mathematics 345
Applied Nonlinear Dynamics and Chaos

Instructor: W. Nagata

Time: $2\frac{1}{2}$ hours

Special instructions:

A non-programmable, non-graphing calculator and one $8\frac{1}{2}'' \times 11''$ page of notes may be used. No other aids are permitted.

Marks

[25] 1. Consider the one-dimensional equation for $\phi(t)$ in the phase circle $S^1 = \mathbf{R}/(2\pi\mathbf{Z})$, given by

$$\dot{\phi} = f(\phi, r) = r\cos\phi - \sin\phi\cos\phi, \qquad (1)$$

where $r \in \mathbf{R}$ is a parameter. (Some or all of the following trigonometric identities may be useful: $\sin(A+B) = \sin A\cos B + \cos A\sin B$, $\cos(A+B) = \cos A\cos B - \sin A\sin B$, $\sin A\cos A = \frac{1}{2}\sin(2A)$, $\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos(2A)$, $\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos(2A)$.)

(a) Verify that $\phi^* = \pi/2 \pmod{2\pi}$ is a fixed point of (1), for all r. For all values of r, $-\infty < r < \infty$, determine the linearized stability of the fixed point ϕ^* and classify it as: hyperbolic and stable, hyperbolic and unstable, or non-hyperbolic.

(b) Show that a pitchfork bifurcation occurs at $\phi^* = \pi/2 \pmod{2\pi}$, $r_c = 1$. Find the normal form of this bifurcation.

(c) Sketch the set $\mathcal{Z} = \{(\phi, r) \in S^1 \times \mathbf{R} : f(\phi, r) = 0\}$.

(d) There is at least one other bifurcation, in addition to the one referred to in part (b). Give all points (ϕ^*, r_c) in $S^1 \times \mathbf{R}$ where there is an additional bifurcation, and state the type (saddle-node, transcritical, or pitchfork). No normal forms are required. (*Hint:* look at your plot of the set $\mathcal{Z}$.)

(e) Sketch the phase portrait, in S^1, of (1) with $r = 0$. Label each fixed point with its ϕ-value.

[25] 2. (a) Consider the two-dimensional system for $(x(t), y(t))$ in the phase plane $\mathbf{R}^2$, given by

$$\dot{x} = y, \qquad \dot{y} = x + x^2. \qquad (2)$$

i. Find all the fixed points of (2) in the phase plane.

ii. Use linear stability analysis to classify each fixed point as hyperbolic or non-hyperbolic. If a fixed point is hyperbolic, determine whether it is an attractor, a repeller, or a saddle point. If a fixed point is non-hyperbolic, determine whether the linearization has pure imaginary eigenvalues, a simple zero eigenvalue, or a double zero eigenvalue.

iii. Find a conserved quantity. You may **either** write down a conserved quantity and then verify that it is indeed conserved, **or** write down the conditions a conserved quantity for this system must satisfy and then construct a function that satisfies these conditions.

iv. Sketch the phase portrait of (2). Indicate the (global) stable manifold and the (global) unstable manifold of any saddle point. Indicate homoclinic orbit(s), if any exist.

(b) Consider the two-dimensional system for $(x(t), y(t))$ in the phase plane $\mathbf{R}^2$, given by

$$\dot{x} = y, \qquad \dot{y} = x + x^2 - \delta y, \tag{3}$$

where δ is a parameter, $\delta > 0$. Note that the fixed points of (3) are the same as those of (2).

i. Find a Liapunov function $V(x, y)$ so that V decreases along all trajectories of (3) except fixed points. Justify your choice of V (*i.e.* calculate $\dot{V} = \frac{d}{dt}V(x(t), y(t))$ along trajectories and explain carefully why V indeed decreases along all trajectories except fixed points).

ii. Sketch the phase portrait of (3), for small (positive) δ. Indicate the (global) stable manifold and the (global) unstable manifold of any saddle point. Also indicate the (global) basin of attraction of any stable fixed point.

[25] 3. Consider the two-dimensional system for $(x(t), y(t))$ in the phase plane $\mathbf{R}^2$, given by

$$\dot{x} = -\alpha x - 5y + x^2 - x^3 - xy^2, \qquad \dot{y} = 5x - \alpha y + xy - x^2 y - y^3, \tag{4}$$

where α is a parameter, $-1 < \alpha < 1$.

(a) Given that the origin $(x^*, y^*) = (0, 0)$ is the unique fixed point of (4) for all α, use linear stability analysis to classify the fixed point, for each α, as hyperbolic or non-hyperbolic. If the fixed point is hyperbolic, determine whether it is an attractor, a repeller, or a saddle point. If the fixed point is non-hyperbolic, determine whether the linearization has pure imaginary eigenvalues, a simple zero eigenvalue, or a double zero eigenvalue.

(b) Show that in polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$ ($r^2 = x^2 + y^2$, $\tan\theta = y/x$, where $r \geq 0$, $\theta \in S^1$), the system (4) transforms to

$$\dot{r} = -\alpha r + r^2 \cos\theta - r^3, \qquad \dot{\theta} = 5.$$

(c) Let $V(x, y) = x^2 + y^2$, and calculate $\dot{V} = \frac{d}{dt}V(x(t), y(t))$ along trajectories. Determine an explicit numerical value of $b > 0$ so that the closed and bounded disk

$$D = \{\ (x, y) \in \mathbf{R}^2 :\ V(x, y) \leq b\,\}$$

is trapping for all α with $-1 < \alpha < 1$.

(d) Find a critical value α_c, $-1 < \alpha_c < 1$, for which there is a Hopf bifurcation for (4). State whether a closed orbit exists for $\alpha < \alpha_c$ or for $\alpha > \alpha_c$, and give justification. You may use the result of part (c) without having answered it.

[25] 4. Consider the one-dimensional map

$$x_{n+1} = f(x_n)$$

for x_n in the phase line $\mathbf{R}$, where

$$f(x) = r - x^2,$$

and r is a parameter, $-\infty < r < \infty$.

(a) For each r, find every fixed point or determine that no fixed point exists. For each r and for each fixed point, use linear stability analysis to determine whether the fixed point is hyperbolic and stable, hyperbolic and unstable, or non-hyperbolic.

(b) Find values of (r, x) for which there is:

i. a saddle-node bifurcation;

ii. a flip bifurcation.

Explain your reasons, but do not attempt to calculate normal forms or period-2 cycles.

(c) It can be shown that for some values of r, there is a period-3 cycle: three distict points $\{ p_1, p_2, p_3 \}$ such that $f(p_1) = p_2$, $f(p_2) = p_3$, and $f(p_3) = p_1$. If f^3 denotes the third iterate of f, *i.e.* $f^3(x) = f(f(f(x)))$, show that

$$f^3(p_1) = p_1,$$

and

$$(f^3)'(p_1) = f'(p_3)f'(p_2)f'(p_1).$$

(d) Do you expect that the map has a chaotic attractor for some values of r? Explain.

Total marks [100]

The End