Final Exam Duration: 150 minutes

*This test has **6 questions** on **13 pages**, for a total of 75 points.*

- Do not turn this page over. You will have 150 minutes for the exam (between 8:30-11:00)
- This is a closed-book examination: no books, notes or electronic devices of any kind.
- You must justify all answers, regardless of the "operative word". Write in complete English sentences; proofs must be clear and concise.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.
- This exam is printed double-sided with the last two pages blank.

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Signature: ______________________

Question:	1	2	3	4	5	6	Total
Points:	18	15	7	5	15	15	75
Score:							

3 marks 1. (a) Define a *subgroup* of a group.

3 marks (b) Let G a group, H, K subgroup of G. Show that the intersection $H \cap K$ is a subgroup of G as well.

3 marks (c) Given two positive integers n, m with $m|n$ construct a group G with a subgroup H of orders n, m respectively.

3 marks

(d) Let $H = \langle r \rangle$ be a cyclic group, and let $g, h \in H$. Show that one of g, h, gh^{-1} is a square in H (an element of the form x^2).

3 marks

(e) Let $n \geq 10$ and let $\alpha \in S_n$ be an element of order 6. What are the possible cycle structures of α ?

3 marks

(f) List the abelian groups of order 56.

2. Let G be a group, $S \subset G$ a subset.

5 marks (a) Let $g \in G$. Show that the subgroup $\langle S \rangle$ generated by S satisfies

$$g \langle S \rangle g^{-1} = \left\langle gSg^{-1} \right\rangle$$

5 marks (b) Suppose that all the elements of S commute with each other. Show that $\langle S \rangle$ is abelian

5 marks (c) Suppose that $G/Z(G)$ is cyclic. Show that $G = Z(G)$ (hint: use part (b))

7 marks 3. Let the group G act transitively on the set X, and let $N \lhd G$ be a normal subgroup. Show that any two N-orbits in X have the same cardinality.

5 marks

4. Let G be a group of order 12. Show that either $G \simeq A_4$ or $G \simeq N \rtimes H$ where N has order 3, H has order 4.

5. Let G be a group of order $987 = 3 \cdot 7 \cdot 47$.

6 marks (a) Show that G has a unique subgroup of order 47, and that this subgroup is central.

3 marks (b) Construct two non-isomorphic groups of order 987.

6 marks (c) Classify the groups of order 987.

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6. For a group G and subsets $A, B \subset G$ let $[A, B]$ denote the subgroup generated by all commutators $[a, b] = aba^{-1}b^{-1}$ where $a \in A, b \in B$.

5 marks

(a) Let $N \triangleleft G$, $A \subset G$. Show that the image of A in G/N is central in G/N if and only if $[G, A] \subset N$.

5 marks

(b) Define a chain of subgroups by $G_0 = G$ and $G_{n+1} = [G, G_n]$ (e.g. $G_1 = [G, G]$ is the derived subgroup, but this is not the derived series). Show that each G_n is normal and that $G_n \supset G_{n+1}$.

5 marks

(c) Suppose that the image of $S \subset G$ by the quotient map generates G/G_1. Show that the image of S generates G/G_n for all n.

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