

**Final Exam Duration: 150 minutes**

*This test has 6 questions on 13 pages, for a total of 75 points.*

- Do not turn this page over. You will have 150 minutes for the exam (between 8:30-11:00)
- This is a closed-book examination: no books, notes or electronic devices of any kind.
- You must justify all answers, regardless of the "operative word". Write in complete English sentences; proofs must be clear and concise.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.
- This exam is printed double-sided with the last two pages blank.

First Name: \_\_\_\_\_ Last Name: \_\_\_\_\_

Student-No: \_\_\_\_\_ Signature: \_\_\_\_\_

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|-----------|----|----|---|---|----|----|-------|
| Question: | 1  | 2  | 3 | 4 | 5  | 6  | Total |
| Points:   | 18 | 15 | 7 | 5 | 15 | 15 | 75    |
| Score:    |    |    |   |   |    |    |       |

### Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
  - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
  - (iii) purposely viewing the written papers of other examination candidates;
  - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

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| 3 marks |
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1. (a) Define a *subgroup* of a group.

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| 3 marks |
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- (b) Let  $G$  a group,  $H, K$  subgroup of  $G$ . Show that the intersection  $H \cap K$  is a subgroup of  $G$  as well.

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| 3 marks |
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- (c) Given two positive integers  $n, m$  with  $m|n$  construct a group  $G$  with a subgroup  $H$  of orders  $n, m$  respectively.

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| 3 marks |
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(d) Let  $H = \langle r \rangle$  be a cyclic group, and let  $g, h \in H$ . Show that one of  $g, h, gh^{-1}$  is a square in  $H$  (an element of the form  $x^2$ ).

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| 3 marks |
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(e) Let  $n \geq 10$  and let  $\alpha \in S_n$  be an element of order 6. What are the possible cycle structures of  $\alpha$ ?

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| 3 marks |
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(f) List the abelian groups of order 56.

2. Let  $G$  be a group,  $S \subset G$  a subset.

5 marks

(a) Let  $g \in G$ . Show that the subgroup  $\langle S \rangle$  generated by  $S$  satisfies

$$g \langle S \rangle g^{-1} = \langle g S g^{-1} \rangle$$

5 marks

(b) Suppose that all the elements of  $S$  commute with each other. Show that  $\langle S \rangle$  is abelian

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| 5 marks |
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(c) Suppose that  $G/Z(G)$  is cyclic. Show that  $G = Z(G)$  (hint: use part (b))

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| 7 marks |
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3. Let the group  $G$  act transitively on the set  $X$ , and let  $N \triangleleft G$  be a normal subgroup. Show that any two  $N$ -orbits in  $X$  have the same cardinality.

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| 5 marks |
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4. Let  $G$  be a group of order 12. Show that either  $G \simeq A_4$  or  $G \simeq N \rtimes H$  where  $N$  has order 3,  $H$  has order 4.

5. Let  $G$  be a group of order  $987 = 3 \cdot 7 \cdot 47$ .

6 marks

(a) Show that  $G$  has a unique subgroup of order 47, and that this subgroup is central.

3 marks

(b) Construct two non-isomorphic groups of order 987.

6 marks

(c) Classify the groups of order 987.



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6. For a group  $G$  and subsets  $A, B \subset G$  let  $[A, B]$  denote the subgroup generated by all commutators  $[a, b] = aba^{-1}b^{-1}$  where  $a \in A, b \in B$ .

5 marks

- (a) Let  $N \triangleleft G$ ,  $A \subset G$ . Show that the image of  $A$  in  $G/N$  is central in  $G/N$  if and only if  $[G, A] \subset N$ .

5 marks

- (b) Define a chain of subgroups by  $G_0 = G$  and  $G_{n+1} = [G, G_n]$  (e.g.  $G_1 = [G, G]$  is the derived subgroup, but this is not the derived series). Show that each  $G_n$  is normal and that  $G_n \supset G_{n+1}$ .

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| 5 marks |
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- (c) Suppose that the image of  $S \subset G$  by the quotient map generates  $G/G_1$ . Show that the image of  $S$  generates  $G/G_n$  for all  $n$ .

*This page has been left blank for your workings and solutions.*

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