

The University of British Columbia
Final Examination - December 2013
Mathematics 320

Time: 2.5 hours

Last Name _____ First _____ Signature _____

Student Number _____

Special Instructions:

No books, notes, or calculators are allowed. Marks depend on quality of proofs. You can use parts of problems to prove other parts without having done the parts you use.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. Let A be a subset of \mathbb{R} which has a lower bound.

(a) (4 points) What is the definition of $\inf A$?

(b) (6 points) Let L be the set of all lower bounds for A . This is a subset of \mathbb{R} . Using the definitions of \inf and \sup show that $\sup L = \inf A$.

2. Let X be a connected metric space.

(a) (4 points) Define what it means for X to be connected.

(b) (6 points) Prove that X is not countable.

3. Let X be a compact metric space and let K be a closed subset of X .

(a) (4 points) According to the definition of compactness, what must you prove in order to show that K is compact?

(b) (6 points) Prove that K is compact. Hint. The complement of a closed set is open.

4. Let $f(x) = \frac{1}{x}$ for $x \in \mathbb{R}$ and $x \neq 0$.

(a) (4 points) What is the ϵ, δ definition that f is continuous at 1?

(b) (6 points) Use your answer in part (a) to show that $f(x)$ is continuous at 1.

5. You must justify your conclusions.

(a) (1 point) Give an example of a divergent series $\sum a_n$ such that $\limsup \frac{|a_{n+1}|}{|a_n|} = 1$.

(b) (2 points) Give an example of a convergent series $\sum a_n$ such that $\limsup \frac{|a_{n+1}|}{|a_n|} = 1$.

(c) (2 points) Can $\sum_{n=1}^{\infty} a_n$ be convergent if $\limsup \left| \frac{a_{n+1}}{a_n} \right| = 2$?

- (d) (5 points) For the series $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ find R such that the series converges for $|x| < R$ and diverges for $|x| > R$.

6. Let $E = \mathbb{Q} \cap [0, 1]$ be the set of rational numbers in the unit interval. For a function $f : E \rightarrow \mathbb{R}$ we say that f has a *continuous extension* g if (i) g is a real valued function defined for all real numbers in $[0, 1]$, (ii) $g(x) = f(x)$ whenever x is in E , (iii) g is continuous at every point in $[0, 1]$.

(a) (2 points) Suppose f has two continuous extensions g and h . Prove that $g(x) = h(x)$ for $x \in [0, 1]$.

(b) (4 points) Define what it means for f to be uniformly continuous on E .

(c) (2 points) Suppose f has a continuous extension g . Show that this implies that f is uniformly continuous.

(d) (2 points) Give an example of a function f which is continuous on E and which has no continuous extension.

7. Let f be a continuous function that maps the unit interval $[0, 1]$ in \mathbb{R} to itself. Assume that f has a derivative f' which is defined and continuous on $[0, 1]$ and that $|f'(x)| < 1$ for $x \in [0, 1]$.

(a) (4 points) State the mean value theorem.

(b) (6 points) Show that there is a constant $M < 1$ such that for all x, y in $[0, 1]$,
 $|f(x) - f(y)| \leq M|x - y|$.

8. You must briefly justify your answers to the following questions. Recall that the Taylor expansion of $f(x)$ in powers of $x - \alpha$ is $\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(\alpha)(x - \alpha)^k$.

(a) (2 points) Are all convergent sequences Cauchy sequences?

(b) (2 points) Are intervals the connected subsets of \mathbb{Q} ?

(c) (2 points) Is the interior of a connected set connected?

- (d) (2 points) Let f be a real valued function with derivatives of all orders, all of which are defined on the interval $[1, 2]$ in \mathbb{R} . Let $P(x)$ be the Taylor polynomial for f of degree $n - 1$ in powers of $x - 1$. In terms of f what are the derivatives of $P(x)$ at $x = 1$?

- (e) (2 points) What are the first three terms of the Taylor expansion in powers of x for the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$