The University of British Columbia

Term 2, April 12, 2016
MATH 318: Probability with Physical Applications
Instructor: B. Kolesnik

Last Name: ____________________ **First Name:** ____________________

Student Number: ____

Make sure the exam has **14 pages** including the cover page.
The exam consists of **9 questions** worth a total of **100 marks**.
The value of each part of each question is given in the margins.
No aids are permitted. There are tables on the last two pages.
Show all work and calculations. Numerical answers need not be simplified.
Duration: **2 hours and 30 minutes**.

Question	Points	Score
1	16	
2	16	
3	7	
4	9	
5	7	
6	12	
7	10	
8	12	
9	11	
Total	100	

1. The parts of this question are not related to each other.

(4 marks) (a) A poker hand is a uniformly random set of 5 cards from a deck of 52 cards, each of which is one of 4 suits (Hearts, Diamonds, Clubs, Spades) and one of 13 face values (A,2,. . .,10,J,Q,K).

Compute (with explanations) the probability that a poker hand contains (exactly) two pairs ($aabbc$, with a,b,c distinct face values).

(4 marks) (b) A fair dice is a cube with faces labelled 1,2,. . . ,6, which when rolled shows one of its faces uniformly at random.

Two fair die are rolled, showing two distinct values. Compute (with explanations) the probability that one of the values is 6.

(4 marks) (c) Recall the Monty Hall scenario: There are Doors #1,2,3. Uniformly at random, a car is placed behind one of the doors and goats are placed behind the other two doors. To begin, the contestant picks one of the doors. The host then opens one of the other doors, revealing a goat. (If the contestant's door conceals a goat, the host opens the other door concealing a goat, whereas if the contestant's door conceals the car, the host opens one of the other two doors uniformly at random.) Finally, the contestant can either open the door they selected initially or switch and open the other (unopened) door, and claim what lies behind their chosen door.

Using the Law of Total Probability, explain why the contestant wins the car with probability $2/3$ by initially picking Door #1 and then switching.

(4 marks) (d) Let $X \sim \text{Geo}(p)$. Show that $E(X) = 1/p$. *Hint:* One way is to use conditional expectation.

2. A magician initially distributes k rabbits and k bouquets into two hats, with each hat containing exactly k items. At each step, the magician's assistant selects an item uniformly at random from each hat and interchanges their locations (that is, the selected item from Hat #1 is put in Hat #2, and the selected item from Hat #2 is put in Hat #1). Let X_n, $n \geq 0$, denote the number of rabbits in Hat #1 after n steps.

(1 marks) (a) Explain why $(X_n)_{n\geq 0}$ is a Markov chain.

(4 marks) (b) Find the state space S and the transition probabilities $P_{i,j}$, for all $i, j \in S$.

(2 marks) (c) Explain why the Markov chain is reversible. (A full proof is not required, but provide a convincing argument.)

(8 marks) (d) Find the stationary distribution π of the Markov chain. (If you determine it by guessing, be sure to fully explain your reasoning and verify that your guess is correct.)

(1 marks) (e) Suppose that Hat #1 initially contains exactly 1 rabbit (and $k - 1$ bouquets). How many steps on average does it take until Hat #1 once again contains exactly 1 rabbit?

(7 marks) 3. An airplane is missing. Based on the airplane's flight plan and the geography of the region, it has been determined that the airplane is equally likely to be in any one of three locations. A search and rescue team will find the airplane in the ith location, if in fact the airplane is in this region, with probability $p_i = 1 - q_i$, for $i = 1, 2, 3$. (The number q_i is called the "overlook probability" since if the airplane is in the ith location, it is overlooked by the team with probability q_i while searching the ith location.)

Find the conditional probability that the airplane is in the ith location (for each $i = 1, 2, 3$) given that the airplane is not found while searching the first ($i = 1$) location.

4. The parts of this question are not related to each other.

(5 marks) (a) A sample of size $n = 16$ is collected. The sample mean is $\bar{X} = 12$ and the sample variance is $S^2 = 1$. Assuming the data are from a normal distribution, find a 95% confidence interval for the true mean μ. Does this sample provide grounds to reject the hypothesis that $\mu = 11$ at a p-value of 0.05?

(4 marks) (b) Suppose that X_1, X_2 are independent normal random variables, with $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, 2$. Find (with full justification) the distribution of $X = X_1 + X_2$ (give the name and parameter(s)).

(7 marks) 5. Suppose that U_1, U_2 are independent uniform random variables on $(0, 1)$, $U_i \sim \text{Unif}(0, 1)$, $i = 1, 2$. Find the variance of their minimum, that is, compute $\text{Var}(M)$, where $M = \min\{U_1, U_2\}$.

6. Consider the Markov chain with state space $S = \{0, 1, \ldots, 5\}$ and transition matrix

$P =$

	0	1	2	3	4	5
0	0	1/2	1/2	0	0	0
1	0	0	0	0	0	1
2	1/4	0	1/4	0	1/2	0
3	0	1	0	0	0	0
4	0	0	0	2/3	0	1/3
5	0	0	0	1	0	0

(4 marks) (a) Draw a transition diagram for the Markov chain with arrows indicating possible transitions and their corresponding probabilities.

(2 marks) (b) Find the communication class(es) of the Markov chain. Is it reducible or irreducible?

(2 marks) (c) Which states are recurrent and which are transient?

(2 marks) (d) Which states are aperiodic and which are periodic (give the period)?

(2 marks) (e) Find (with explanations) the long run proportion of the time that the Markov chain spends in state 1.

7. Robins and blackbirds make short visits to a bird feeder. The number of robins seen by time t is a Poisson process $(R_t)_{t\geq 0}$ with rate α. The number of blackbirds seen by time t is a Poisson process $(B_t)_{t\geq 0}$ with rate β. Moreover, these two Poisson processes are independent.

   (You may use the following facts: Suppose that X_1, X_2 are independent with $X_i \sim \mathrm{Exp}(\lambda_i)$, $i = 1, 2$. Then $\min\{X_1, X_2\} \sim \mathrm{Exp}(\lambda_1 + \lambda_2)$ and $P(X_1 < X_2) = \lambda_1/(\lambda_1 + \lambda_2)$.)

(1 marks) (a) Explain what it means for $(R_t)_{t\geq 0}$ to be a Poisson process with rate α.

(2 marks) (b) On average, how long does it take for n robins in total to have been seen at the feeder?

(3 marks) (c) Let $T_t = R_t + B_t$ denote the total number of birds seen by time t. Explain why $(T_t)_{t\geq 0}$ is a Poisson process, and give the rate of the process.

(4 marks) (d) Given that n birds have arrived by time t, find and identify the conditional distribution of the number of robins that have arrived by time t. (Give the name of the distribution and its parameter(s).)

(4 marks) 8. (a) State the Central Limit Theorem.

(8 marks) (b) One hundred numbers are rounded to the nearest integer and then added together. Assuming roundoffs (that is, differences between rounded and unrounded numbers) are independent and identically distributed as $\mathrm{Unif}(-1/2, 1/2)$ random variables, use the Central Limit Theorem to estimate the probability that the sum of the rounded numbers and the sum of unrounded numbers differ by at most 1.

9. The parts of this question are not related to each other.

(4 marks) (a) Let i, j be two states in the same communication class of a Markov chain. Suppose that state i is recurrent. Show that state j is also recurrent.

(7 marks) (b) Consider a Markov chain with state space $S = \{0, 1, 2, 3\}$ which always transitions to state 3 when in state 0, and when in state i, $i = 1, 2, 3$, it transitions to one of the states $0, 1, \ldots i - 1$ uniformly at random. Find the long run proportion of time the Markov chain is in state 0.

Table 1: Common Distributions

Distribution	Mean	Variance	Characteristic function
Binomial: $\text{Bin}(n,p)$	np	$np(1-p)$	$(1-p+pe^{it})^n$
Geometric: $\text{Geo}(p)$	$1/p$	$\frac{1-p}{p^2}$	$\frac{pe^{it}}{1-(1-p)e^{it}}$
Poisson: $\text{Poi}(\lambda)$	λ	λ	$e^{\lambda(e^{it}-1)}$
Uniform: $\text{Unif}(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{ita}-e^{itb}}{it(b-a)}$
Exponential: $\text{Exp}(\lambda)$	$1/\lambda$	$1/\lambda^2$	$\frac{\lambda}{\lambda-it}$
Normal: $N(\mu,\sigma^2)$	μ	σ^2	$e^{i\mu t-\sigma^2t^2/2}$

Table 2: Cumulative distribution function $\Phi(x)$ of standard Normal distribution

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Table 3: Critical values of Student's T-distribution

1 Tail	0.10	0.05	0.025	0.01	0.005	0.001	
2 Tails	0.20	0.10	0.05	0.02	0.01	0.002	DFs
	3.078	6.314	12.71	31.82	63.66	318.3	1
	1.886	2.920	4.303	6.965	9.925	22.33	2
	1.638	2.353	3.182	4.541	5.841	10.21	3
	1.533	2.132	2.776	3.747	4.604	7.173	4
	1.476	2.015	2.571	3.365	4.032	5.893	5
	1.440	1.943	2.447	3.143	3.707	5.208	6
	1.415	1.895	2.365	2.998	3.499	4.785	7
	1.397	1.860	2.306	2.896	3.355	4.501	8
	1.383	1.833	2.262	2.821	3.250	4.297	9
	1.372	1.812	2.228	2.764	3.169	4.144	10
	1.363	1.796	2.201	2.718	3.106	4.025	11
	1.356	1.782	2.179	2.681	3.055	3.930	12
	1.350	1.771	2.160	2.650	3.012	3.852	13
	1.345	1.761	2.145	2.624	2.977	3.787	14
	1.341	1.753	2.131	2.602	2.947	3.733	15
	1.337	1.746	2.120	2.583	2.921	3.686	16
	1.333	1.740	2.110	2.567	2.898	3.646	17
	1.330	1.734	2.101	2.552	2.878	3.610	18
	1.328	1.729	2.093	2.539	2.861	3.579	19
	1.325	1.725	2.086	2.528	2.845	3.552	20
	1.323	1.721	2.080	2.518	2.831	3.527	21
	1.321	1.717	2.074	2.508	2.819	3.505	22
	1.319	1.714	2.069	2.500	2.807	3.485	23
	1.318	1.711	2.064	2.492	2.797	3.467	24
	1.316	1.708	2.060	2.485	2.787	3.450	25
	1.315	1.706	2.056	2.479	2.779	3.435	26
	1.314	1.703	2.052	2.473	2.771	3.421	27
	1.313	1.701	2.048	2.467	2.763	3.408	28
	1.311	1.699	2.045	2.462	2.756	3.396	29
	1.310	1.697	2.042	2.457	2.750	3.385	30