

## Final Exam

December 8, 2008

15:30 – 18:00

**No books. No notes. No calculators. No electronic devices of any kind.**  
**The duration of this exam is 150 minutes.**  
**There are 10 problems, each worth an equal number of points.**

**Problem 1.** (6 points)

This problem is about the logarithmic spiral in the plane

$$\vec{r}(t) = e^t \langle \cos t, \sin t \rangle, \quad t \in \mathbb{R}.$$

- (a) Find the arc length of the piece of this spiral which is contained in the unit circle.
- (b) Reparametrize the logarithmic spiral with respect to arc length, measured from  $t = -\infty$ .

**Problem 2.** (6 points)

Find the point in the first quadrant where the graph of the function  $y = \frac{1}{3}x^3$  has maximal curvature.

**Problem 3.** (6 points)

Under the influence of a force field  $\vec{F}$ , a particle of mass 2 kg is moving with constant speed 3 m/s along the path given as the intersection of the plane  $z = x$  and the parabolic cylinder  $z = y^2$ , in the direction of increasing  $y$ . Find  $\vec{F}$  at the point  $(1, 1, 1)$ . (Length is measured in m along the three coordinate axes.)

**Problem 4.** (6 points)

The curve  $C$  is the helix which winds around the cylinder  $x^2 + y^2 = 1$  (counterclockwise, as viewed from the positive  $z$ -axis, looking down on the  $xy$ -plane). It starts at the point  $(1, 0, 0)$ , winds around the cylinder once, and ends at the point  $(1, 0, 1)$ . Compute the line integral of the vector field

$$\vec{F}(x, y, z) = \langle -y, x, z^2 \rangle$$

along  $C$ .

**Problem 5.** (6 points)

Evaluate the line integral  $\int_C \vec{F} \cdot d\vec{r}$ , where  $\vec{F}$  is the conservative vector field

$$\vec{F}(x, y, z) = \langle y + ze^x, x + e^y \sin z, z + e^x + e^y \cos z \rangle$$

and  $C$  is the curve given by the parametrization

$$\vec{r}(t) = \langle t, e^t, \sin t \rangle, \quad t \text{ from } 0 \text{ to } \pi.$$

**Problem 6.** (6 points)

Let  $S$  be the surface of a cone of height  $a$  and base radius  $a$ . The surface  $S$  does not include the base of the cone or the interior of the cone. Find the centre of mass of  $S$ .

Locate the cone in a coordinate system so that its base is in the  $xy$ -plane, and its vertex on the  $z$ -axis. So the vertex will be the point  $(0, 0, a)$ . The base is a circle of radius  $a$  in the  $xy$ -plane with centre at the origin. The cone surface is characterized by the fact that for every point of  $S$ , the distance from the  $z$ -axis and the distance from the  $xy$ -plane add up to  $a$ .

Use polar coordinates. Note that because of the symmetry of the cone, only the  $z$ -component of the centre of mass requires an integral to be calculated.

**Problem 7.** (6 points)

A surface  $S$  is given by the parametric equations

$$\begin{aligned}x &= 2u^2 \\y &= v^2 \\z &= u^2 + v^3\end{aligned}$$

Find an equation for the tangent plane to  $S$  at the point  $(8, 1, 5)$ .

**Problem 8.** (6 points)

Find the flux of the vector field  $\langle x + y, x + z, y + z \rangle$  through the cylindrical surface whose equation is  $x^2 + z^2 = 4$ , and which extends from  $y = 0$  to  $y = 3$ . (Only the curved part of the cylinder is included, not the two disks bounding it on the left and right.) The orientation of the surface is outward, i.e., pointing away from the  $y$ -axis.

**Problem 9.** (6 points)

A simple closed curve  $C$  lies in the plane  $x + y + z = 1$ . The surface this curve  $C$  surrounds inside the plane  $x + y + z = 1$  has area 5. The curve  $C$  is oriented in a clockwise direction as observed from the positive  $z$ -axis looking down at the plane  $x + y + z = 1$ .

Compute the line integral of  $\vec{F}(x, y, z) = \langle z^2, x^2, y^2 \rangle$  around  $C$ .

**Problem 10.** (6 points)

The surface  $S$  is the part above the  $xy$ -plane of the surface obtained by revolving the graph of  $z = 1 - x^4$  around the  $z$ -axis. The surface  $S$  is oriented such that the normal vector has positive  $z$ -component. The circle with radius 1 and centre at the origin in the  $xy$ -plane is the boundary of  $S$ .

Find the flux of the divergenceless vector field  $\vec{F}(x, y, z) = \langle yz, x + z, x^2 + y^2 \rangle$  through  $S$ .