Final Exam

Friday, December 15, 2006.

No books, notes or calculators

Problem 1. (10 points)
(The scalar field is $\mathbb{R}$.) Consider the matrix A and column vector $\vec{b}$:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(a) (6 points) Find the $PA = LU$ factorization for A.

(b) (2 points) Recall that the $PA = LU$ factorization can be used to rewrite the system $A\vec{x} = \vec{b}$ as two systems with triangular coefficient matrix. Write down these two triangular systems.

(c) (2 points) Solve the two triangular systems to find all $\vec{x}$ such that $A\vec{x} = \vec{b}$.

Problem 2. (10 points)
(The scalar field is $\mathbb{R}$.) Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 & 4 \\ 2 & 4 & 3 & 0 & 5 \\ 1 & 2 & -1 & 5 & 5 \end{pmatrix}$$

(a) Find a basis for the nullspace of A.

(b) Find a basis for the column space of A.

(c) Find a basis for the left nullspace of A.

(d) Find a basis for the row space of A.

Problem 3. (10 points)
Consider the discrete dynamical system

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$$

which describes the evolution of the vector $\begin{pmatrix} a & b & c \end{pmatrix}^T$.

(a) (2 points) Explain what a Markov matrix (stochastic matrix) is and why the coefficient matrix of this discrete dynamical system is a Markov matrix.

(b) (6 points) Find a fixed vector for this discrete dynamical system, i.e., a vector $\begin{pmatrix} a & b & c \end{pmatrix}^T$ such that

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix},$$

for all n.

(c) (2 points) Find the limit of the vector $\begin{pmatrix} a_n & b_n & c_n \end{pmatrix}^T$ as $n \to \infty$ when the initial vector is

$$\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 39 \end{pmatrix}.$$

Problem 4. (15 points)
Consider the real matrix

$$A = \begin{pmatrix} 2 & 1 \\ -8 & -2 \end{pmatrix}$$

(a) (7 points) use complex numbers to diagonalize A, this means to find matrices S, Λ, and S^{-1}, such that

$$A = S\Lambda S^{-1},$$

where Λ is diagonal.

(b) (2 points) Write down (using complex numbers) the general solution of the system of differential equations

$$\frac{d}{dt}\vec{x}(t) = A\,\vec{x}(t).$$

(c) (2 points) Discuss the long term behaviour of the solutions of $\vec{x}\,' = A\,\vec{x}$.

(d) (2 points) Solve the initial value problem

$$x_1'(t) = 2x_1(t) + x_2(t) \quad x_2'(t) = -8x_1(t) - 2x_2(t)\,; \qquad x_1(0) = 3 \quad x_2(0) = -2\,.$$

(e) (2 points) Write down a (real!) basis for the real vector space of all differentiable functions $\vec{x}(t) : \mathbb{R} \to \mathbb{R}^2$, satisfying $\vec{x}\,'(t) = A\,x(t)$, for all t.

Problem 5. (8 points)
Consider the real matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) (2 points) Explain what property the matrix A has, which assures that you can diagonalize it without the help of complex numbers.

(b) (6 points) find a real matrix S and a diagonal matrix D, such that

$$A = SDS^T \,.$$

Problem 6. (12 points)
Consider the real matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) (6 points) Find the QR factorization of A.

(b) (3 points) Find the least-squares 'solution' to $A\,\vec{x} = \vec{b}$, where

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} \,.$$

(c) (3 points) Find the projection of $\vec{b}$ onto the column space of A.

Problem 7. (12 points)
Decide whether or not the following statements are true or false. If true, give a proof. If false, give a counterexample.

(a) The product of two orthogonal matrices is orthogonal.

(b) If $AB = I$, then $BA = I$. (Warning: do *not* assume that A and B are square matrices.)

(c) For every 8×5 matrix A,

$$\dim N(A) \geq 3 \,.$$

(d) For every 5×5 matrix A with real entries,

$$\det(A^T A) \geq 0 \,.$$

(e) If $A\,\vec{x} = \vec{b}$ and $A^T\,\vec{y} = \vec{0}$, then $\vec{y}^{\,T}\,\vec{b} = 0$.

Problem 8. (8 points)
Let

$$A = \begin{pmatrix} 0 & 1 & 1 & x \\ 1 & 0 & x & 1 \\ 1 & x & 0 & 1 \\ x & 1 & 1 & 0 \end{pmatrix}$$

(a) Calculate the determinant $\det A$.

(b) Find all x such that $\det A = 0$.