The University of British Columbia
Final Examination - April 21, 2005

Mathematics 307

Section 201-202
Instructors: Drs. Carrell and Purbhoo

Closed book examination — Time: 2.5 hours

Name ______________________ **Signature** ______________________

Student Number ______________

Special Instructions:
- Be sure that this examination has 12 pages. Write your name on top of each page.
- No calculators or notes are permitted.
- For maximum credit show all your work.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate should be prepared to produce her/his library/AMS card upon request.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of examination.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Making use of any books, papers, or memoranda, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates.
- Smoking is not permitted during examinations.

1		15
2		20
3		15
4		20
5		15
6		15
7		10
8		20
9		20
Total		150

[15pt] 1. Consider the symmetric matrix

$$B = \begin{pmatrix} 2 & -2 & 0 & 4 \\ -2 & 3 & 2 & 1 \\ 0 & 2 & 3 & 3 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

[10] (a) Find the LDU decomposition of B.

[5] (b) Find the number of positive and negative eigenvalues of B.

[20pt] 2. Let V denote the row space of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 5 & -2 \end{pmatrix}$$

[5] (a) Find bases of V and $V^{\perp}$.

[5] (b) Find the matrix of the projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ of $\mathbb{R}^3$ onto V.

[5] (c) Find the orthogonal sum decomposition $(1,\ 1,\ -1)^T = \mathbf{v} + \mathbf{w}$ where $\mathbf{v} \in V$ and $\mathbf{w} \in V^{\perp}$.

[5] (d) Find the distance from $(1,\ 1,\ -1)^T$ to V.

[15pt] 3. Let P be a real symmetric $n \times n$ matrix such that $P^2 = P$. Also, let $R = I_n - 2P$.

[5] (a) Show that R is an orthogonal matrix and that $R^2 = I_n$.

[5] (b) If P is the matrix of the projection of $\mathbb{R}^n$ onto a subspace V and if Q is the matrix of the projection of $\mathbb{R}^n$ onto the orthogonal complement $V^\perp$, explain (either algebraically or geometrically) why $PQ = O$ and $RQ = Q$.

[5] (c) Find the eigenvalues of R and give a description of the corresponding eigenspaces.

[20pt] 4. Consider the following 3×6 matrix over the field $\mathbb{F}_2$:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

[5] (a) Find a basis for the null space $\mathcal{N}(A)$.

[5] (b) Let C be the binary code consisting of all 6-bit strings $\mathbf{c}$ so that $\mathbf{c}^T$ is in $\mathcal{N}(A)$. Find the minimal distance $d(C)$ of C.

[5] (c) Find the unique codeword nearest 000101.

[5] (d) Find an example of a vector in $(\mathbb{F}_2)^6$ which does not have a unique nearest codeword.

[15pt] 5. Find A^N for any positive integer N when A is the matrix

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}.$$

[15pt] 6. Let $\mathcal{B}$ be the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$, where

$$\mathbf{v}_1 = (1,0,0)^T, \quad \mathbf{v}_2 = (1,1,0)^T, \quad \mathbf{v}_3 = (1,1,1)^T,$$

and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$\mathcal{M}_{\mathcal{B}}^{\mathcal{B}}(T) = \begin{pmatrix} 2 & -3 & 3 \\ -2 & 1 & -7 \\ 5 & -5 & 7 \end{pmatrix}.$$

[7] (a) Find $T(\mathbf{v}_3)$.

[8] (b) Calculate the matrix of T with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$. You may leave your answer unsimplified.

[10pt] 7. Determine whether or not the following two matrices are similar:

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

[20pt] 8. Answer the following questions by filling in the blanks to get a correct mathematical theorem, statement or definition.

(a) Every $n \times n$ matrix A over the complex numbers can be expressed in the form $A = UTU^{-1}$, where T is ____________________ and U is unitary.

(b) Every Hermitian matrix can be expressed in the form $A = PRP^{-1}$, where P is ____________ and R is a ____________ diagonal matrix.

(c) Every real $m \times n$ matrix A of rank ____ can be expressed in the form $A = QR$, where Q has orthogonal ____________, R is upper triangular and the determinant of R is ________________.

(d) If all eigenvalues of an $n \times n$ complex matrix A are zero, then $A^n =$________.

(e) A complex matrix A is normal if and only if ____________________.

(f) If A is a real matrix, then the column space of A is the orthogonal complement to the ____________ space of ________.

9. True or False: two points for the right answer and 2 more points for also giving a correct reason.

(a) Every permutation matrix is diagonalizable.

(b) The matrix $\begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & -1 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}$ is positive definite.

(c) If $A\mathbf{x} = \mathbf{b}$ for some $\mathbf{x}$ and $A^T\mathbf{y} = 0$, then $\mathbf{y}^T\mathbf{b} = 0$.

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(d) Every matrix similar to a Hermitian matrix is normal.

(e) If P is a positive definite matrix, then all entries of P are non-negative.

Scratchwork