

Final Exam

- This exam consists of **7 questions** on **11 pages** worth a total of 75 points.
- Make sure this exam has **11 pages**.
- Duration: 2.5 hours.
- Enter your final answer in the box when provided.
- Calculators, notes, or other aids are not allowed to use.
- Explain your solution thoroughly, and justify all answers. No credit might be given for unsupported answers.
- The exponential functions appearing in the answers should not be evaluated.
- The last page is empty for additional space. If a question is solved there, mark it clearly or it may not be graded.

First Name: _____ Last Name: _____

Student No.: _____ Signature: _____

Student conduct during examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	75	

15 marks

1. (a) Define period for a state of a discrete time Markov chain.

(b) Define the no-memory property for a random variable X .

(c) Define the transition probability function $P_{ij}(t)$ for continuous time Markov chains.

(d) Give a sufficient condition for the existence of the limiting probabilities of a discrete time Markov chain.

(e) Set up the detailed balance equations for a continuous time Markov chain.

10 marks

2. Capa plays either one or two chess games every day, with the number of games that she plays on successive days being a Markov chain with transition probabilities

$$P_{1,1} = 0.2, \quad P_{1,2} = 0.8, \quad P_{2,1} = 0.4, \quad P_{2,2} = 0.6.$$

Capa wins each game with probability p . Suppose she plays two games on Monday.

- (a) (2 marks) What is the probability that she wins exactly one game on Tuesday?

Answer:

- (b) (3 marks) What is the expected number of games that she plays on Wednesday?

Answer:

(c) (5 marks) In the long run, on what proportion of days does Capa win all her games?

Answer:

10 marks

3. Consider the random walk on the integers $\{0, 1, 2, 3\}$ which takes steps $+1$ (to the right) with probability $\frac{1}{4}$ and -1 (to the left) with probability $\frac{3}{4}$, except at the endpoints where there is reflection; this means that a step from 1 to 0 is always followed by a step from 0 to 1, and a step from 2 to 3 is always followed by a step from 3 to 2.

(a) (3 points) Determine the probability transition matrix for this Markov chain.

Answer:

(b) (7 points) What fraction of time does it spend in state 0?

Answer:

10 marks

4. Customers arrive at a shop according to a Poisson process of with rate 2 per a minute. Assume that a customer is woman with probability $\frac{3}{4}$ and man with probability $\frac{1}{4}$ independently of the other customers.

(a) (3 marks) What is the probability that at least two men arrive in a 6 minute period?

Answer:

(b) (3 marks) What is the probability that exactly two men and four women arrive in a 4 minute period?

Answer:

(c) (4 marks) From 9:20-10:20am there were 50 customers in total. What is the probability that none of them arrived between 9:30-9:34am?

Answer:

10 marks

5. Let $0 \leq p \leq \frac{1}{3}$. Consider a branching process where the number X of offspring of an individual is zero with probability p , one with probability $2p$, and two with probability $1 - 3p$. Initially there is one individual.

(a) (3 marks) For what values of p will the branching process become extinct after a finite number of generations with probability 1? Explain.

Answer:

(b) (5 marks) For the case $p = \frac{1}{5}$, calculate the probability that the branching process survives forever.

Answer:

- (c) (2 marks) Assume again that $p = \frac{1}{5}$. Given that there are three individuals in the second generation, find the probability that the branching process survives forever.

Answer:

10 marks

6. Particles are emitted by a radioactive substance according to a Poisson process of rate 6. Each particle exists for an exponentially distributed time, of rate 2, independent of the other particles, before disappearing. Let $X(t)$ be the number of particles existing at time t . This is a birth and death process.

(a) (3 marks) Determine the birth and death rates.

Answer:

- (b) (7 marks) Find the limiting probabilities P_j . Name the limiting distribution and its parameter(s).

Answer:

10 marks

7. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean $\frac{1}{4}$ hour. What is the average number of customers in the shop?

Answer:

This page left blank for your workings and solutions. Problems solved: _____