Final Exam

- This exam consists of **7 questions** on **11 pages** worth a total of 75 points.
- Make sure this exam has **11 pages**.
- Duration: 2.5 hours.
- Enter your final answer in the box when provided.
- Calculators, notes, or other aids are not allowed to use.
- Explain your solution thoroughly, and justify all answers. No credit might be given for unsupported answers.
- The exponential functions appearing in the answers should not be evaluated.
- The last page is empty for additional space. If a question is solved there, mark it clearly or it may not be graded.

First Name: ____________________ Last Name: ____________________

Student No.: ____________________ Signature: ____________________

Question	Points	Score
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	75	

15 marks 1. (a) Define period for a state of a discrete time Markov chain.

(b) Define the no-memory property for a random variable X.

(c) Define the transition probability function $P_{ij}(t)$ for continuous time Markov chains.

(d) Give a sufficient condition for the existence of the limiting probabilities of a discrete time Markov chain.

(e) Set up the detailed balance equations for a continuous time Markov chain.

10 marks

2. Capa plays either one or two chess games every day, with the number of games that she plays on successive days being a Markov chain with transition probabilities

$$P_{1,1} = 0.2, \quad P_{1,2} = 0.8, \quad P_{2,1} = 0.4, \quad P_{2,2} = 0.6.$$

Capa wins each game with probability p. Suppose she plays two games on Monday.

(a) (2 marks) What is the probability that she wins exactly one game on Tuesday?

Answer:

(b) (3 marks) What is the expected number of games that she plays on Wednesday?

Answer:

(c) (5 marks) In the long run, on what proportion of days does Capa win all her games?

Answer:

10 marks

3. Consider the random walk on the integers $\{0, 1, 2, 3\}$ which takes steps $+1$ (to the right) with probability $\frac{1}{4}$ and -1 (to the left) with probability $\frac{3}{4}$, except at the endpoints where there is reflection; this means that a step from 1 to 0 is always followed by a step from 0 to 1, and a step from 2 to 3 is always followed by a step from 3 to 2.

   (a) (3 points) Determine the probability transition matrix for this Markov chain.

   Answer:

   (b) (7 points) What fraction of time does it spend in state 0?

   Answer:

10 marks

4. Customers arrive at a shop according to a Poisson process of with rate 2 per a minute. Assume that a customer is woman with probability $\frac{3}{4}$ and man with probability $\frac{1}{4}$ independently of the other customers.

   (a) (3 marks) What is the probability that at least two men arrive in a 6 minute period?

   Answer:

   (b) (3 marks) What is the probability that exactly two men and four women arrive in a 4 minute period?

   Answer:

   (c) (4 marks) From 9:20-10:20am there were 50 customers in total. What is the probability that none of them arrived between 9:30-9:34am?

   Answer:

10 marks

5. Let $0 \le p \le \frac{1}{3}$. Consider a branching process where the number X of offspring of an individual is zero with probability p, one with probability $2p$, and two with probability $1 - 3p$. Initially there is one individual.

   (a) (3 marks) For what values of p will the branching process become extinct after a finite number of generations with probability 1? Explain.

   Answer:

   (b) (5 marks) For the case $p = \frac{1}{5}$, calculate the probability that the branching process survives forever.

   Answer:

(c) (2 marks) Assume again that $p = \frac{1}{5}$. Given that there are three individuals in the second generation, find the probability that the branching process survives forever.

Answer:

10 marks

6. Particles are emitted by a radioactive substance according to a Poisson process of rate 6. Each particle exists for an exponentially distributed time, of rate 2, independent of the other particles, before disappearing. Let $X(t)$ be the number of particles existing at time t. This is a birth and death process.

   (a) (3 marks) Determine the birth and death rates.

   Answer:

   (b) (7 marks) Find the limiting probabilities P_j. Name the limiting distribution and its parameter(s).

   Answer:

10 marks

7. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean $\frac{1}{4}$ hour. What is the average number of customers in the shop?

Answer:

This page left blank for your workings and solutions. Problems solved: ______