Be sure this exam has 9 pages including the cover

The University of British Columbia

Sessional Exams – 2011 Term 2
Mathematics 303 Introduction to Stochastic Processes
Dr. D. Brydges

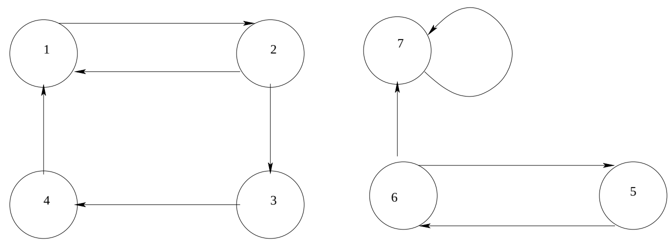
Last Name: ______________________ **First Name:** ______________________

Student Number: ____

This exam consists of **8** questions worth **10** marks each so that the total is **80**. No aids are permitted.
Please show all work and calculations. *On some parts no explanation will mean no marks.*
Numerical answers need not be simplified.

Problem	total possible	score
1.	10	
2.	10	
3.	10	
4.	10	
5.	10	
6.	10	
7.	10	
8.	10	
total	80	

1. Consider the Markov chain $(X_n,\ n = 1, 2, ..)$ whose seven states and transitions of positive probability are as indicated in the diagram.



(3 points) (a) What are the communicating classes?

(1 points) (b) Does this Markov chain have a stationary distribution? Briefly explain.

(1 points) (c) Suppose $X_0 = 1$. Does $\lim_{n\to\infty} P^n_{i,j}$ exist when $i = 1$? Briefly explain.

(3 points) (d) Define the period of a state and use your definition to determine the periods of the recurrent states.

(2 points) (e) Suppose $\mathbb{P}(X_0 = i) = \rho_i$ for every state i where ρ_i, $i = 1, 2, \ldots, 7$ are given probabilities. Give a formula for $\mathbb{P}(X_1 = 7)$ in terms of ρ_i and the transition probability matrix P_{ij}.

2. A soap opera has N characters. In each episode some are evil and the others are good. Before making each episode the director tosses a coin which has probability p of showing heads. If the result is heads then a character is chosen at random and, if evil, becomes good, if good, remains good; if the result is tails then a character is chosen at random and, if good, becomes evil, if evil, remains evil. For $n = 1, 2, \ldots$, let X_n denote the number of good characters in episode n. For $n = 0$ set $X_n = 0$.

(2 points) (a) Draw the transition diagram of the Markov chain $(X_n)_{n\geq 0}$ when $N = 4$, showing the transitions of positive probability. You need not insert the probabilities.

(3 points) (b) Find the nonzero transition probabilities P_{ij}.

(5 points) (c) This soap opera has run for a very long time. Find, as a formula involving the number of characters N, the proportion of episodes in which there are exactly two good characters. What is the answer for $N = 2$?

3. An urn contains four balls, which can be black or white. At each time $n = 0, 1, 2, \ldots$ a ball is chosen at random from the urn and replaced by a ball of the opposite colour.

(3 points) (a) What does it mean to say that a state is positively recurrent? Are all the states in this Markov chain positively recurrent?

(2 points) (b) For this Markov chain, running forever, all four balls are white for $\frac{1}{16}$ of the times. If we start the sytem in the state where all the balls are white, what is the expected time before they are again all white?

(5 points) (c) Initially there is one white ball and three black balls. What is the probability that they all become white before they are all black? (Hint. Gamblers ruin.)

4. Let S_n be the position at time n of symmetric simple random walk on $\mathbb{Z}$ with $S_0 = 0$.

(4 points) (a) Thinking of steps to the left and the right what is the formula for $\mathbb{P}(S_{2n} = 0)$ in terms of n?

(2 points) (b) Let $N = \#\{n : S_n = 0\}$ be the number of visits to the origin. State and prove the formula that relates $\mathbb{E}N$ to $\mathbb{P}(S_n = 0)$. Hint. Indicator functions.

(2 points) (c) Define what it means for state 0 to be recurrent. What is the relation to $\mathbb{E}N$?

(2 points) (d) Prove that the origin is recurrent. ($n! \sim \sqrt{2\pi n} n^n e^{-n}$).

5. Bacteria reproduce by cell division. In a unit of time, a bacterium will either die (with probability $\frac{1}{4}$), stay the same (with probability $\frac{1}{4}$), or split into 2 bacteria (with probability $\frac{2}{4}$). Interpret this as a branching process. Let Z_n be the number of bacteria at time $n = 0, 1, 2, \ldots$ and let $G_n(s)$ be the generating function for Z_n.

(2 points) (a) Suppose initially there is only one bacterium. Find $G_1(s)$ as a function of s.

(4 points) (b) Suppose initially there is only one bacterium. Find $G_2(s)$ as a function of s.

(2 points) (c) Suppose that initially there are 100 bacteria. What is the expected number in generation n.

(2 points) (d) Suppose that initially there are 100 bacteria. What is the probability that the population goes extinct.

(5 points) 6. (a) One of the standard definitions of a Poisson process $N(t)$ of rate r, involves a statement about $N(s+h) - N(s)$ when h is small. Write out this definition explaining the meaning of terms in it such as increment and $o(h)$.

(2 points) (b) Suppose that crimes in a large city occur according to a rate r Poisson process $N(t)$. For each crime, the criminal responsible for the crime is sent to prison with probability 1/5. Ignoring the delay between the crime and the beginning of the prison sentence, what process $P(t)$ describes the arrivals of new prisoners? What assumption not yet mentioned are you making?

(3 points) (c) Partly justify your answer in part (b) by showing that $P(t)$ satisfies a $N(s+h) - N(s)$ axiom as in part (a).

7. Blackbirds come to a bird feeder according to a Poisson process $B(t)$ with rate b per hour; robins come to the same bird feeder according to an independent Poisson process $R(t)$ with rate r per hour.

(4 points) (a) What is the probability that exactly four birds arrive between 12 and 12:30pm?

(2 points) (b) Exactly one bird arrived during the first hour. What is the probability that it actually arrived during the first quarter of an hour.

(2 points) (c) Suppose that the first robin arrives at time S_1 and the second robin arrives at time S_2. What is the probability that $S_2 > 4S_1$?

(2 points) (d) What is the probability that exactly one blackbird arrives before the first robin? ($\int_0^\infty t^n e^{-at}\, dt = n! a^{-n-1}$.)

8. Students arrive at a help centre according to a rate r Poisson process. When there are $n \geq 1$ students in the centre, the first one to leave does so at a random Exp $(2r)$ time.

(2 points) (a) Let X_t be the number of students in the centre at continuous time t. Regarding this as a birth/death process, what are the parameters λ_n and μ_n for $n = 0$ and for $n = 1, 2, \ldots$?

(2 points) (b) For $n = 0, 1, 2, \ldots$, if there are presently n students in the centre, what is the expected time to the next transition?

(2 points) (c) For $n = 0, 1, 2, \ldots$, if there are presently n students in the centre, what is the probability that the next transition will be to $n + 1$ students?

(4 points) (d) Suppose that there are presently no students in the centre. What is the expected time until there are two students?