

# MATH 303 – Sect 201 – 2007/2008

## README

- Fill this cover-page with your name and student number.
- Prepare to produce your ID upon request.
- Use the present sheets **ONLY**. Calculators, books, formula sheets and collaborations are **NOT** allowed. (In case of violation your work will be **CONFISCATED**).

**This exam has 6 problems**  
(and some blank pages)

**Name:**

**Number:**

P. 1:

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Tot:

**Problem 1**

A Markov chain has space state  $\{1, 2, 3, 4, 5, 6\}$  and transition matrix

$$P = \begin{pmatrix} 0 & 0 & 3/4 & 0 & 1/4 & 0 \\ 0 & 2/5 & 0 & 0 & 1/5 & 2/5 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 2/7 & 0 & 5/7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- a) Find the communicating classes.
- b) For each state of the chain, write, with a short explanation, if it is transient, null recurrent or positive recurrent.
- c) For each state of the chain, write, with a short explanation, the period.

### Problem 2

Henry has three houses, one in Vancouver, one in Berlin and one in Prague, and he likes to spend some years in each of them. He may change place in the end of the year, and decides his destination according to the following rule:

- if he has spent one year in Vancouver, for the next year he is equally likely to move to Berlin or Prague if cheap flights to Europe are available; otherwise he has to remain in Vancouver one more year;
- if he has spent a year in Berlin, he moves to Vancouver if there is a cheap flight to North America, otherwise moves to Prague;
- if he has spent one year in Prague, he is equally likely to spend the next year in Prague or Berlin.

Suppose that, in the end of any given year, a cheap flight to Europe is available with probability  $2/3$ , and a cheap flight to North America is available with probability  $1/2$ . And let  $X_n$  be the city in which Henry sojourns in the  $n$ -th year:  $\{X_n\}$  is a Markov chain.

- a) Find the state-space and the transition probability of  $\{X_n\}$ .
- b) Compute the long-run proportion of years that Henry spends in Berlin or Prague.
- c) Compute the long-run proportion of times that Henry moves from Berlin to Prague.

**Problem 3**

In a population of bacteria, each individual lives one day, then either dies with probability  $1/4$  or splits into two offsprings. Suppose that on the 31st of December 2007 the population of bacteria is made of one individual.

- a) *Find the expected number of individuals in the population on the 3rd of January 2008.*
- b) *Find the probability that the population will eventually die out.*
- c) *Find the probability that the population will eventually die out, given that on the 6th of January 2008 the population is made of 2 individuals.*

**Problem 4**

The service-times of server  $S_1$  and server  $S_2$  are random variables with exponential distribution and rates  $\lambda_1$  and  $\lambda_2$  respectively. When the customer  $C$  arrives,  $S_1$  is working on customer  $A$  and  $S_2$  is working on  $B$ ; therefore  $C$  has to wait until it is processed by the first server that becomes free.

- a) *Compute the probability that  $B$  is still in service when the service for  $C$  has been completed.*
- b) *Compute the cumulative distribution function (c.d.f.) of the random time that  $C$  has to wait before being serviced.*
- c) *Compute the cumulative distribution function (c.d.f.) of the random time during which  $C$  is in service.*
- d) *Compute the expected time that  $C$  spends in the system (waiting for the service and being serviced).*

### Problem 5

The weather in Vancouver can be sunny, cloudy or rainy, and can change because of “troughs”, elongated areas of low pressure. Suppose that the number of arrivals of troughs is a Poisson process with parameter  $\lambda = 1/3$ : with probability  $p = 1/3$  the trough is small, otherwise is large. Then we have:

- when it is sunny, the arrival of a small trough changes the weather into cloudy; the arrival of a large trough into rainy;
- when it is cloudy, the arrival of any kind of trough changes the weather into rainy; but if there is no arrival by time  $T$ , exponential random variable of rate  $\mu = 2/9$ , the weather becomes sunny;
- when it is rainy, there is no new trough (the pressure is already low), and after a time  $T_2$ , exponential random variable with rate  $\nu_2 = 1/6$ , the weather becomes sunny.

Let  $X(t)$  be the state of the weather at time  $t$ :  $\{X(t)\}$  is a continuous time Markov chain.

- a) Find the parameters of the embedded Markov chain for  $\{X(t)\}$  (state space, probability of transition in one jump, rates of the waiting times).
- b) Compute the long time proportion of time that it is rainy.

**Problem 6**

A shop is open to the customers from 9am to 7pm. The number of arrivals of women is a Poisson process with an average of 10 per hour; and the number of arrivals of men is an independent Poisson process with an average of 8 per hour.

- a) *Compute the probability that the first two customers arriving after 10am are women.*
- b) *Find the probability mass function of the number of customers (men or women) that arrive between 9am and 10am.*

Math 303. 25 April 2008. Final.



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