The University of British Columbia

Final Examination - 6th January , 2013

Mathematics 300, Section 101, Instructor: Christian Sadel

Closed book examination Time: 2.5 hours

Last Name ____________________ **First** ____________________

Student Number ________________ **Signature** ____________________

Special Instructions:

Explain your reasoning carefully. You will be graded on the clarity of your explanations as well as on the correctness of your answers. Place your student ID on the table.

All electronic devices (including cell phones) **must** be shut off and put away!

No books, notes, or calculators are allowed.

Rules governing examinations

• Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

• Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.

• Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

• Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

(a) speaking or communicating with other candidates, unless otherwise authorized;

(b) purposely exposing written papers to the view of other candidates or imaging devices;

(c) purposely viewing the written papers of other candidates;

(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

• Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

• Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

• Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		12
2		12
3		12
4		18
5		10
6		12
7		10
8		14
Total		100

[12] **1**. Express all of the following terms in the form $a + ib$ with a and b real and simplify the expressions! (Just writing $f = \operatorname{Re} f + i \operatorname{Im} f$ won't give any points!)

(a) $\cos[i \log(-1 + i)]$

(b) $\operatorname{Log}\left(\frac{\sqrt{3}+i}{\sqrt{2}(1+i)}\right)$

(c) All solutions of $\cosh(z) = \frac{1}{2}$.

[12] **2.** Let $f(z) = x + 2y + i(2x - y)^2$, where $z = x + iy$.

(a) Where is $f(z)$ differentiable in the complex plane?

(b) Where is $f(z)$ analytic?

(c) Find all entire functions $g(z)$ that satisfy $\text{Re}(g(z)) = \text{Re}(f(z))$.

[12] **3.** Find a branch $f(z)$ of $(z-1)^{-1/2}$ that is analytic except for $z \in (1 - i\mathbb{R}_+) = \{z : \mathrm{Re}(z) = 1 \text{ and } \mathrm{Im}(z) \leq 0\}$.
Let Γ be the semi-circle centered at 1, going counter-clockwise from 2 to 0. Calculate $\displaystyle\int_\Gamma f(z)\,dz$ for your choice of f.

[18] **4.** Here, $C_r(w)$ denotes the circle of radius r, centered at w, traversed once in counter-clockwise direction. Calculate the following.

[3] (a) $\displaystyle\oint_{C_1(i)} \bar{z}\,dz$

[5] (b) $\displaystyle\oint_{C_{50}(2)} z\cos(\tfrac{1}{z})\,dz$

[5] (c) $\displaystyle\oint_{C_5(0)} \frac{100!\,e^{iz}}{(z+1)^{100}}\,dz$

[5] (d) $\displaystyle\oint_{C_4(0)} \frac{z\cos(z)}{(2z-\pi)^2}\,dz$

[10] **5.** Let f be an entire function and assume that there exists $C, R > 0$ such that one has $|f(z)| < C|z|^n$ for all $|z| > R$. Show that f is a polynomial of degree smaller or equal to n. (Hint: Taylor series!)

[12] **6.** Let

$$f(z) = \frac{1}{(3z-1)(z+2)}$$

(a) Find a Laurent series valid for large $|z|$.

(b) Expand $f(z)$ in a Laurent series in an annular region centered at 0 which contains $z = 1$. Give the region of convergence.

(c) Find all the residues of f.

(d) Let Γ be the closed polygonal path (path consistent of straight lines) going from $(3+i)$ to $(-3-i)$ to $(-3+i)$ to $(3-i)$ and back to $(3+i)$. Calculate $\int_\Gamma f(z)\, dz$.

[10] **7.** By formally multiplying Laurent-series, express the residue of the product $e^{1/z} \cdot \frac{1}{1-z}$ at 0 as an infinite sum and then calculate this sum.

[14] **8.** Evaluate the following integrals. Simplify if possible.

(a) $I_a = \displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2 + 2x + 2)(x^2 + 1)}\, dx$

(b) $I_b = \displaystyle\int_{0}^{2\pi} \frac{1}{a + \sin(\theta)}\, d\theta$ where $a > 1$.

**** THE END ****