

The University of British Columbia
Final Examination - December 7, 2005

Mathematics 265

All Sections

Closed book examination

Time: 2.5 hours

Special Instructions:

- Be sure that this examination booklet has 4 pages.
- No calculators or notes other than your one page formula sheet are permitted.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate should be prepared to produce her/his library/AMS card upon request.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of examination.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
 - (a) Making use of any books, papers, or memoranda, other than those authorized by the examiners.
 - (b) Speaking or communicating with other candidates.
 - (c) Purposely exposing written papers to the view of other candidates.

1. [10] Consider the ODE $y' = y(2 - y)$.

(a) Draw a direction field for this ODE.

(b) Without solving the equation, determine $\lim_{t \rightarrow \infty} y(t)$ if $y(t)$ is a solution of this ODE with

i. $y(0) = 0$,

ii. $y(0) = 2$,

iii. $y(0) = -1$.

2. [10] Consider the ODE

$$(t - 1) \frac{dy}{dt} = \frac{2t}{t + 1} + 1,$$

Find the solution of this differential equation that satisfies $y(0) = 0$. Determine the largest interval in which your solution is valid.

3. [20] Consider the initial value problem

$$y'' + 4y = g(t) \quad y(0) = 3, \quad y'(0) = 1.$$

(a) Solve the corresponding homogeneous equation, and prove that the two solutions you have found are a fundamental set of solutions.

(b) For each of the following cases, what form would you guess for a particular solution $Y(t)$, if you were going to use the method of undetermined coefficients? **Do not solve the equation!**

i. $g(t) = t^2 e^t$

ii. $g(t) = e^t \sin t + t$

iii. $g(t) = \cos(2t)$

(c) Solve the initial value problem when $g(t) = 5 \cos(t)$.

4. [15] Let $f(t) = \sin(t)$ and $g(t) = e^{-2t}$.
- (a) Compute $W(f, g)(t)$.
 - (b) Are f and g linearly independent on $(0, 2\pi)$? Justify your answer.
 - (c) Can one find an ODE $y'' + p(t)y' + q(t)y = 0$ where p and q are continuous on $(0, 2\pi)$ for which both f and g are solutions? Explain.
 - (d) Compute $h(t) = f \star g$ where “ \star ” denotes convolution.

5. [15] Solve the initial value problem

$$y'' + 3y' + 2y = (t - 2)u_2(t), \quad y(0) = y'(0) = 0.$$

What is $\lim_{t \rightarrow \infty} y(t)$?

6. [15] Consider the system of first-order ODEs given by

$$\begin{aligned}x_1' &= -3x_1 + 4x_2 + 4e^{-t} \\x_2' &= -x_1 + 2x_2 + 3e^{-t}\end{aligned}$$

- (a) Write this system as a single matrix equation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$.
- (b) Find the general solution of the homogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
- (c) Sketch the trajectory on the x_1x_2 -plane passing through the point $(x_1, x_2) = (4, 2)$. Specify a point on the x_1x_2 -plane so that the trajectory passing through this point approaches 0 as $t \rightarrow \infty$.
- (d) Now, solve the non-homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g}(t)$ with initial conditions $x_1(0) = 2$ and $x_2(0) = 4$.

7. [15] Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Laplace Transform Table

$f(t)$	$F(s) = \mathbb{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s - a}, \quad s > a$
$\sin(at)$	$\frac{a}{s^2 + a^2}, \quad s > 0$
$\cos(at)$	$\frac{s}{s^2 + a^2}, \quad s > 0$