

Math 257/316 Final Exam, April 2013

Last Name: _____ First Name: _____

Student Number: _____ Signature: _____

Instructions. The exam lasts 2.5 hours. **No calculators or electronic devices of any kind are permitted.** A formula sheet is attached. There are **12 pages** in this test including this cover page, blank pages, and the formula sheet. Unless otherwise indicated, show all your work.

Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (a) speaking or communicating with other candidates, unless otherwise authorized;
 - (b) purposely exposing written papers to the view of other candidates or imaging devices;
 - (c) purposely viewing the written papers of other candidates;
 - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem #	Value	Grade
1	20	
2	24	
3	20	
4	20	
5	16	
Total	100	

1. [20 marks] For the ODE initial value problem

$$(1+x)y'' + y = 0, \quad y(0) = 2, \quad y'(0) = 3,$$

find the first 5 non-zero terms of the series solution about $x = 0$.

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2. For the non-homogeneous heat equation problem

$$\begin{aligned}u_t &= u_{xx} + e^x & 0 < x < 1, \quad t > 0 \\u(0, t) &= 0, \quad u(1, t) = 1 \\u(x, 0) &= f(x),\end{aligned}$$

- (a) [8 marks] Find the steady-state solution.
- (b) [8 marks] Find the full solution $u(x, t)$, expressing any coefficients in terms of integrals (which you need not evaluate).
- (c) [8 marks] Briefly describe how you would use the method of finite differences to find an approximate solution to this problem. Use the notation $u_n^k = u(x_n, t_k)$ to denote the values of u on the finite difference mesh, and include how you propose to incorporate the boundary and initial conditions. *In case it is useful, the Taylor expansion formula is $f(x + \Delta x) = f(x) + f'(x)\Delta x + \frac{1}{2}f''(x)(\Delta x)^2 + O((\Delta x)^3)$.*

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3. [20 marks] Use the method of separation of variables and Fourier series to find the solution $u(x, t)$ of the following problem involving a damped wave equation with Neumann BCs:

$$\begin{aligned}u_{tt} + u_t &= u_{xx} & 0 < x < \pi, \quad t > 0 \\u_x(0, t) &= 0, \quad u_x(\pi, t) = 0 \\u(x, 0) &= x, \quad u_t(x, 0) = 1 + \cos(x).\end{aligned}$$

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4. Let f be the following function defined on $[0, \pi/2]$:

$$f(\theta) = \begin{cases} 1 & 0 \leq \theta < \pi/4 \\ 0 & \pi/4 \leq \theta \leq \pi/2 \end{cases}$$

- (a) [10 marks] Sketch both the odd and even π -periodic extensions of f , find both its Fourier sine and Fourier cosine series, and find the value each series converges to for each $\theta \in [0, \pi/2]$.
- (b) [10 marks] Find the solution $u(r, \theta)$ of the following problem for Laplace's equation in a $1/4$ -disk:

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0 \quad 0 < r < 1, \quad 0 < \theta < \pi/2, \\ u_{\theta}(r, 0) &= 0, \quad u_{\theta}(r, \pi/2) = 0 \\ u &\text{ bounded as } r \rightarrow 0, \quad u(1, \theta) = f(\theta), \end{aligned}$$

with $f(\theta)$ the function above.

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5. (a) [10 marks] Find the eigenvalues λ and eigenfunctions $X(x)$ of the following Sturm-Liouville eigenvalue problem on $[1, 2]$:

$$(x^2 X')' + \lambda X = 0, \quad X'(1) = X'(2) = 0.$$

Hint: to save time, you may consider only the cases $\lambda = 0$ and $\lambda > 1/4$.

- (b) [6 marks] Find the solution $u(x, t)$ of the following variable-coefficient heat equation problem,

$$u_t = (x^2 u_x)_x, \quad 1 < x < 2, \quad t > 0,$$

$$u_x(1, t) = 0, \quad u_x(2, t) = 0, \quad u(x, 0) = f(x),$$

expressing any coefficients in terms of integrals (which you need not evaluate).

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