

1. Consider the differential equation

$$6x^2y'' - xy' + 2(1+x)y = 0 \quad (1)$$

- (a) Classify the points $0 \leq x < \infty$ as ordinary points, regular singular points, or irregular singular points.
- (b) Find two values of r such that there are solutions of the form $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$.
- (c) Use the series expansion in (b) to determine two independent solutions of (1). You only need to calculate the first three non-zero terms in each case.

[20 marks]

(Question 1 Continued)

(Question 1 Continued)

2. Consider the following initial boundary value problem for the heat equation:

$$\begin{aligned}u_t &= u_{xx} - u, & 0 < x < 1, & \quad t > 0 \\u(0, t) &= 0, & u_x(1, t) &= 1 \\u(x, 0) &= 1\end{aligned}\tag{2}$$

- (a) Determine a steady state solution to the boundary value problem. [4 marks]
- (b) Use this steady state solution to determine the solution to the boundary value problem (2) by separation of variables.

HINT: The following integral may be useful:

$$\int_0^1 \sinh(x) \sin(\beta x) dx = \frac{1}{\beta^2 + 1} (\sin \beta \cosh 1 - \beta \cos \beta \sinh 1)$$

[8 marks]

- (c) Briefly describe how you would use the method of finite differences to obtain an approximate solution this boundary value problem that is accurate to $O(\Delta x^2, \Delta t)$ terms. Use the notation $u_n^k \simeq u(x_n, t_k)$ to represent the nodal values on the finite difference mesh. Explain how you propose to approximate the boundary condition $u_x(1, t) = 1$ with $O(\Delta x^2)$ accuracy.

[8 marks]

[total 20 marks]

(Question 2 Continued)

(Question 2 Continued)

3. Consider the following initial-boundary value problem:

$$u_{tt} + 2\gamma u_t = c^2 u_{xx}, \quad 0 < x < 1, \quad t > 0$$

$$u(0, t) = u(1, t) = 0$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0.$$

- (a) Assuming that $0 < \gamma < \pi c$, use separation of variables to determine the solution to this boundary value problem. [12 marks]
- (b) If $\gamma = 0$, $c = 1$, and if the initial displacement $f(x) = \sin(4\pi x)$, sketch the shape of the string at time $t = 1/4$. [8 marks]
- [total 20 marks]

(Question 3 Continued)

(Question 3 Continued)

- (a) Consider the eigenvalue problem

$$\begin{aligned}r^2 R'' + rR' + \lambda R &= 0 \\ R(1) &= 0 = R(2)\end{aligned}$$

Reduce this problem to the form of a Sturm-Liouville eigenvalue problem. Determine the eigenvalues and corresponding eigenfunctions. [8 marks]

- (b) Use the eigenfunctions in (a) to solve the following mixed boundary value problem for the annular region:

$$\begin{aligned}u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 1 < r < 2, \quad 0 < \theta < \pi \\ u(r, 0) &= 0 \quad \text{and} \quad \frac{\partial u}{\partial \theta}(r, \pi) = f(r) \\ u(1, \theta) &= 0 \quad \text{and} \quad u(2, \theta) = 0\end{aligned}$$

[12 marks]
[total 20 marks]

(Question 4 Continued)

(Question 4 Continued)

4. Solve the inhomogeneous heat conduction problem:

$$\begin{aligned}u_t &= u_{xx} + e^{-t}(1-x), \quad 0 < x < 1, \quad t > 0 \\u(0, t) &= 1, \quad \text{and } u(1, t) = e^{-t} \\u(x, 0) &= 1.\end{aligned}$$

[20 marks]

(Question 5 Continued)

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