The University of British Columbia
Final Examination - April 12, 2006

Mathematics 257

Section 201
Instructor: A. Khadra

Closed book examination | Time: 2.5 hours

Name ______________________ **Signature** ______________________

Student Number ______________

Special Instructions:
- Be sure that this examination has 15 pages. Write your name on top of each page.
- A formula sheet is provided. No programmable/graphing calculators or notes are permitted.
- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate should be prepared to produce her/his library/AMS card upon request.
- No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of examination.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Making use of any books, papers, or memoranda, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates.
- Smoking is not permitted during examinations.

1		20
2		20
3		20
4		10
5		20
6		10
Total		100

[20] 1. Consider the following second order linear homogeneous differential equation, given by

$$2xy'' - y' + 2y = 0 \tag{1}$$

(a) Prove that the point $x_0 = 0$ is a regular singular point for equation (1).

(b) Apply Frobenius' method to find the first four terms of the two linearly independent series solutions for equation (1). What is the general solution?

(c) Is your result is part (b) consistent with the values of the indicial roots obtained? Explain.

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[20] 2. Seek a solution of the form $u(x,t) = X(x)T(t)$ to solve the damped wave equation given by

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 0, \ \dfrac{\partial u}{\partial t}(x,0) = 10. \end{cases}$$

What is the behaviour of $u(x,t)$ as $t \to \infty$?

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[20] 3. Solve the following Poisson problem given by

$$\begin{cases} \nabla^2 u = xy \\ u(0,y) = u(1,y) = 0 \\ u(x,0) = 0,\ u(x,1) = x. \end{cases}$$

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[10] 4. Consider the following Sturm-Liouville problem, given by

$$xy'' + y' + \Big[-\frac{4}{x} + \lambda^2 x \Big] y = 0, \quad 0 < x < 1, \; y(0) \text{ is finite}, \;\; y(1) = 0. \tag{2}$$

(a) Write (2) in the standard Sturm-Liouville form and determine if it is a regular or singular Sturm-Liouville problem.

(b) Determine the eigenvalues and the eigenfunctions of equation (2).

(c) Express the orthogonality relations between two eigenfunctions corresponding to two different eigenvalues obtained in part (b).

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[20] 5. Apply the method of separation of variables to solve the BVP given by

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \\ u(r,0) = u(r,\frac{\pi}{2}) = 0 \\ \frac{\partial u}{\partial r}(1,\theta) = \theta. \end{cases}$$

(Hint: If k is the separation constant, then, for $k \leq 0$, the trivial solution will be generated.)

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[10] 6. The solution to the IBVP

$$\begin{cases} u_t = c^2(u_{rr} + \frac{1}{r}u_r), & 0 < r < a, \quad t > 0 \\ u(a,t) = 0, \quad t > 0, \\ u(r,0) = f(r), \quad 0 < r < a, \end{cases} \tag{3}$$

is given by

$$u(r,t) = \sum_{n=1}^{\infty} A_n e^{-c^2\lambda_n^2 t} J_0(\lambda_n r)$$

with

$$A_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a r f(r) J_0(\lambda_n r) dr,$$

where $\lambda_n = \alpha_n / a$ and α_n is the nth positive zero of J_0.

(a) Solve equation (3) for $a = c = 1$ and $f(r) = 100$.

(b) Is the problem described by equation (3) radially symmetric?

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List of Formulae

- Integrals:

$$\int x\cos ax\,dx = \frac{1}{a}x\sin ax + \frac{1}{a^2}\cos ax + c \quad (a \neq 0).$$
$$\int x\sin ax\,dx = -\frac{1}{a}x\cos ax + \frac{1}{a^2}\sin ax + c \quad (a \neq 0).$$

- If

$$f(x) = a_0 + \sum_{n=1}^{\infty}\Big[a_n\cos(\frac{n\pi}{L}x) + b_n\sin(\frac{n\pi}{L}x)\Big],$$

then

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)dx, \quad n \geq 1,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)dx, \quad n \geq 1.$$

- If

$$u(x,y) = \sum_{n=1}^{\infty}\beta_n\sin(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y),$$

then

$$\beta_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f_2(x)\sin\frac{n\pi}{a}x\,dx.$$

- If $\Lambda_{mn} = (m\pi/a)^2 + (n\pi/b)^2$ and

$$u(x,y) = \sum_{m,n=1}^{\infty} E_{mn}\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y),$$

then

$$E_{mn} = -\frac{4}{ab\Lambda_{mn}}\int_0^b\int_0^a f(x,y)\sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{b}y)dxdy.$$

- If y_m and y_n are two eigenfunctions corresponding to two different eigenvalues, then

$$\int_a^b r(x)y_m(x)y_n(x)dx = 0.$$

List of Properties and Identities for Bessel Functions

Let m be a nonnegative integer and ν be a nonnegative real number.

1. $J_{-m}(x) = (-1)^m J_m(x)$.
2. $J_m(-x) = (-1)^m J_m(x)$. i.e., $J_m(x)$ is an even function when m is even and it is an odd function when m is odd.
3. $$J_m(0) = \begin{cases} 0 & m > 0 \\ 1 & m = 0. \end{cases}$$
4. $\lim_{x \to 0^+} Y_m(x) = -\infty$.
5. $xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x)$.
6. $xJ'_\nu(x) = -\nu J_\nu(x) + xJ_{\nu-1}(x)$.
7. $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)$.
8. $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$.
9. $\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x)$.
10. $\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)$.
11. $\int x^{\nu+1} J_\nu(x) dx = x^{\nu+1} J_{\nu+1}(x) + c$.
12. $\int x^{-\nu+1} J_\nu(x) dx = -x^{-\nu+1} J_{\nu-1}(x) + c$.