Math 256 Section 202 Final Exam
Spring 2005
Instructor: PC Chang

Last Name: ______________________________

First Name: ______________________________

Student Number: ______________________________

Email Address: ______________________________

INSTRUCTIONS:

- Write your last name, first name, student number, and email address in the spaces above.
- No calculators allowed.
- 3 *handwritten* cheat sheets (both sides) allowed.
- This exam consists of 10 questions on 17 pages (including this one).
- The maximum score on this exam is 100.
- You have 180 minutes to complete this exam.
- Good Luck!

1) Answer the following questions. You need not show work for this section.

A) What is the y-intercept of $y = 3x + 1$? (1 mark)

B) True or False: $\int_0^{\infty} \sin x dx$ is an improper integral. (1 mark)

C) Find the general solution of $\frac{dy}{dx} = xy^{-1}$. (1 mark)

D) What is the canonical form of the subcritical pitchfork bifurcation? (1 mark)

E) Find the steady-state solution of $u_t = u_{xx}$ subject to the boundary conditions $u(0,t) = 0$, $u(1,t) = 2$. (1 mark)

F) What is the name of this equation: $u_{tt} = c^2 u_{xx}$? (1 mark)

G) True or False: An example of a linear differential operator is the operator L defined by $Lu \equiv \sin(u')$. (1 mark)

H) True or False: If $(r, \vec{\xi})$ is an eigenvalue-eigenvector pair of a real matrix A, then so is their complex conjugates $(r^*, \vec{\xi}^*)$. (1 mark)

I) True or False: The conditions $y(\alpha) = y_0, y(\beta) = y_1$ for the ODE $y'' + p(x)y' + q(x)y = g(x)$ are known as Dirichlet boundary conditions. (1 mark)

J) True or False: The determinant of a Fundamental Matrix Solution is 0. (1 mark)

2) Solve the homogeneous equation $xdy = 2(x + y)dx$.
(10 marks)

3) Solve $y' + 2xy = e^{-x^2}$, $y(0) = 1$. (10 marks)

4) Find the general solution of the system

$$\frac{dx}{dt} = 2x + 2y,$$
$$\frac{dy}{dt} = x + 3y.$$

with initial conditions $x(0) = 5, y(0) = -1.$ (10 marks)

5) Using variation of parameters, find the general solution of

$$y'' + 3y' + 2y = \left(1 + e^{x}\right)^{-1}.$$

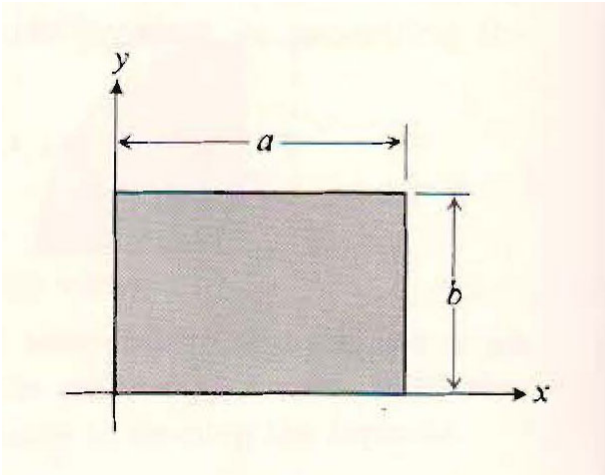
(10 marks)

5 Cont’d)

6a) Consider the autonomous ODE $\frac{dx}{dt} = x\left(r - e^x\right)$. Sketch all the qualitatively different phase portraits that occur as r is varied. Classify the bifurcation that occurs, and find the bifurcation point. (7 marks)

6b) Suppose initially $x(0) = 1$. For what values of r does $\lim_{t \to \infty} x(t) = 0$? (3 marks)

7) Consider a very long (infinite) heat-conducting bar of rectangular cross-section, as shown below.



If the face $y = b$ is kept at a constant temperature T_0, and the other three faces are kept at zero temperature, compute the steady-state temperature distribution. (10 marks)

7 Cont'd)

7 Cont’d)

8a) Consider the autonomous ODE $\frac{d\theta}{dt} = r + \sin\theta$. Sketch all the qualitatively different phase portraits that occur as r is varied. Classify the bifurcations that occur, and find the bifurcation point. (7 marks)

8b) Suppose initially $\theta(0) = \pi$. For what values of r does $\lim_{t\to\infty} \theta(t) = \infty$? (3 marks)

9) Consider the following BVP for the one dimensional heat equation

$$u_t = u_{xx}$$

where $0 < x < 1$ and $t > 0$, with conditions

$$u(0,t) = u(1,t) = 0 \text{ for } t > 0,$$
$$u(x,0) = e^{-x} \text{ for } 0 < x < 1.$$

9a) Give a brief physical interpretation of this problem. (2 marks)

9b) Solve this BVP. (8 marks)

9b Cont’d)

9b Cont’d)

10) A stretched string of length L with its ends fixed at $x = 0$ and $x = L$ has initial profile $u(x,0) = f(x)$ and is initially at rest. For $t > 0$, it is subjected to forced vibrations described by the PDE

$$u_{xx} - c^{-2}u_{tt} = -g''(x),$$

where g is a given function which satisfies $g(0) = g(L) = 0$. One method for solving this problem is to decompose the solution as $u(x,t) = v(x,t) + w(x)$, where v and w each solve a modified version of this problem. Determine these modified problems. In particular, state the differential equations which v and w should solve, and the conditions which v and w should obey. *Do not solve these problems*.

(10 marks)