THIS EXAMINATION CONSISTS OF 8 QUESTIONS. PLEASE CHECK TO ENSURE THAT THIS PAPER IS COMPLETE.

THE UNIVERSITY OF BRITISH COLUMBIA
Term 2 Examinations - April 2005

MATHEMATICS 227
Section 201

TIME: 2.5 Hours

INSTRUCTIONS: No notes, books or calculators are to be used. No credit will be given for the correct answer without the (correct) accompanying work.

1. Let $z=f(x,y)$ be a C^3 function on an open set $D \subset R^2$, and assume that all partial derivatives of orders one, two and three equal zero at a point $(a,b) \in D$ except for $\frac{\partial^3 z}{\partial x \partial y^2}(a,b)=6$. Determine whether (a,b) is a local maximum, a local minimum or a saddle point, and give reasons for your answer. Remark: Your reasons do not need to be completely rigorous but they should be at least heuristically convincing. [10%]

2. Use the method of Lagrange multipliers (no credit will be given for any other method) to find the points on the ellipse $x^2+4y^2=4$ which are closest to the point $(1,0)$. Hint: Minimize the **square** of the distance from a point on the ellipse to $(1,0)$, and be careful to find all four solutions to the equations specified by the method of Lagrange multipliers. [15%]

3. Use the change of variable $x=u/v$ and $y=v$ to evaluate the double integral $\iint_D y\,dx\,dy$ where $D \subset R^2$ is the region specified by the inequalities $y \le 3$, $y \ge x$ and $1 \le xy \le 4$. [15%]

4. Let $\mathbf{F}=(e^x\cos(\pi y^2)+ay-1, bye^x\sin(\pi y^2)+x+y)$ be a vector field in R^2, where a and b are constants.
   (a) Find values of a and b such that $\mathbf{F}$ is conservative. [5%]
   (b) With the values of a and b determined in part (a) evaluate the vector line integral $\int \mathbf{F}\cdot d\mathbf{r}$ taken over any smooth curve starting at (0,0) and ending at (1,1). [5%]

5. Find the surface area of the part of the paraboloid $z=2-x^2-y^2$ lying above the xy-plane. [10%]

6. Let $\mathbf{F}=\dfrac{\mathbf{r}}{\|\mathbf{r}\|^3}$ be a vector field in R^3, where $\mathbf{r}=(x,y,z)$. If $D\subset R^3$ is an open set containing the origin with a smooth boundary ∂D, then it was shown in class that the Divergence or Gauss's Theorem does not apply directly for $\mathbf{F}$, D and ∂D. That is, $\iint_{\partial D} \mathbf{F}\cdot d\mathbf{S} \neq \iiint_D \text{div}(\mathbf{F})dV$, where the the triple integral must be interpreted as an improper triple integral since $\mathbf{F}$ is not defined at the origin.
   (a) Let $\mathbf{G}=\dfrac{\mathbf{r}}{\|\mathbf{r}\|^2}$ with D as above. Does the Divergence Theorem apply directly for $\mathbf{G}$, D and ∂D? More precisely, is $\iint_{\partial D} \mathbf{G}\cdot d\mathbf{S}=\iiint_D \text{div}(\mathbf{G})dV$?
   Here, as above, the triple integral must be interpreted as an improper triple integral since $\mathbf{G}$ is not defined at the origin. Give precise reasons for your answer. [10%]
   (b) Verify that your answer to part (a) is correct in the special case where $D=\{(x,y,z):x^2+y^2+z^2<a^2\}$. [5%]

7. Verify that Stokes's Theorem is true in the special case of the vector field $\mathbf{F}=(y,xz,x^2)$ over the triangle with vertices at (1,0,0), (0,1,0) and (0,0,1). [15%]

8. Let $\omega=\omega_{(x,y)}=xydx$ be a 1-form on R^2. According to the discussion in your textbook, the exterior derivative or differential of ω is a 2-form on R^2 defined by the formula $d\omega=d\omega_{(x,y)}=d(xy)\wedge dx=(ydx+xdy)\wedge dx=xdy\wedge dx=-xdx\wedge dy$. At a point (x,y) $d\omega$ operates on pairs of vectors in R^2. For example, at the point (x,y)=(2,1), $d\omega_{(2,1)}(\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix})=-2dx\wedge dy(\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix})=-2\det\begin{bmatrix}2&0\\3&1\end{bmatrix}=-2(2)=-4$. The purpose of this problem is to ask you to give a more geometric definition of the exterior derivative motivated by the generalized Stokes's Theorem. In order to keep everything very concrete, give a geometric definition of $d\omega_{(2,1)}(\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix})$ involving a limit as $h\to 0$ and an integral of ω over a certain closed curve C_h in R^2. Then verify by use of your definition that $d\omega_{(2,1)}(\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix})=-4$. Your discussion should be accompanied by a picture showing all the key aspects of your solution. [10%]