The University of British Columbia

Math 220 — Introduction to Mathematical Proof

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First Name: ____________________ **Last Name:** ____________________

Student id. ____________________ **Section (circle):** **201** **202**

Instructions

- This exam consists of **10 questions** worth a total of 100 points.
- Justify all answers, show all work and calculations. Extra paper is available as needed.
- No calculators or other aids are permitted.
- Make sure this exam has 12 pages excluding this cover page.
- Duration: 2 hours 30 minutes.

Good luck, and enjoy your summer.

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	8	10	5	5	5	15	12	15	15	100
Score:											

(10 marks) 1. (a) Let $f : A \to B$ be a function and let $C \subseteq A$ and $D \subseteq B$. Define the sets $f^{-1}(D)$ and $f(C)$.

(b) Define the supremum of a set S of real numbers.

(c) State the converse of the statement

"If I will win the lottery then I will buy a new car."

(d) Write the negation of the statement

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{N} \text{ s.t. } x^2 + y < x + y^2.$$

(e) Define what it means for a function $f : A \to B$ to be injective.

(f) Define what it means for the sequence $\{a_n\}$ to converge to a number L.

(g) Define what it means for the series $\sum_{n=1}^{\infty} a_n$ to converge.

(h) Define what it means for a set S to be countable.

(i) State the principle of mathematical induction.

(j) Let A be the set $\{1, x, y\}$. What is the power set $\mathcal{P}(A)$?

(8 marks) 2. For each of the following subsets of $\mathbb{R}$ write its supremum and infimum if they exist. If they do not exist write "None". You do not need to prove your answers.

(a) $\left\{x \in \mathbb{R} \text{ s.t. } -1 < x < 5\right\}$

(b) $\left\{x \in \mathbb{Q} \text{ s.t. } 3 \le x^2 \le 7\right\}$

(c) $\displaystyle\bigcap_{n=1}^{\infty} \left[2 + \frac{1}{n}, 6 - \frac{2}{n}\right]$

(d) $\displaystyle\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, n\right]$

(10 marks) 3. **Using the definition** of convergence for sequences, prove that

(a) $$\lim_{n\to\infty} \frac{n^2+3n+1}{n^2} = 1.$$ **(b)** $$\lim_{n\to\infty} \frac{1-2\cos(n)}{n} = 0.$$

(5 marks) 4. Prove that

$$P \implies (Q \vee R) \equiv (P \implies Q) \vee (P \implies R).$$

(5 marks) 5. Prove that the function $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}$ given by $f(x) = \dfrac{2x}{x-1}$ is bijective.

(5 marks) 6. Let $n \in \mathbb{Z}$. Prove that $n \equiv 3 \pmod{5}$ if and only if $3n + 1$ is divisible by 5.

(15 marks) 7. (a) Let $\{a_n\}$ be a sequence of real numbers defined by

$$a_1 = 1 \qquad \text{and} \qquad a_{n+1} = 2a_n + 1 \qquad \text{for each } n \in \mathbb{N}.$$

Prove that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

(b) Let $\{b_n\}$ be a sequence defined by

$$b_1 = 2 \qquad \text{and} \qquad b_{n+1} = \frac{b_n + \sqrt{b_n}}{2} \qquad \text{for each } n \in \mathbb{N}.$$

Prove that $1 \leq b_n \leq 2$ for each $n \in \mathbb{N}$.

(c) Prove the sequence $\{b_n\}$ above satisfies $b_{n+1} \leq b_n$ for each $n \in \mathbb{N}$.

(12 marks) 8. Let $P \subset \mathbb{N}$ be the set of prime numbers $P = \{2, 3, 5, 7, \ldots\}$. Determine whether the following statements are true or false. Prove your answers ("true" or "false" is not sufficient).

(a) $\forall m \in P, \forall n \in P, m + n \in P$.

(b) $\forall m \in P, \exists n \in P$ s.t. $m + n \in P$.

(c) $\exists m \in P$ s.t. $\forall n \in P, m + n \in P$.

(d) $\exists m \in P$ s.t. $\exists n \in P$ s.t. $m + n \in P$.

(15 marks) 9. Let $f : A \to B$ and $g : B \to A$ be functions so that $g \circ f$ is a surjective function.

(a) Prove that g is surjective.

(b) Give an example of sets A, B and functions f, g as above such that f is not surjective.

(c) Prove or disprove that $f \circ g$ must be surjective.

(15 marks) 10. (a) Let $A \subset \mathbb{R}$ be a bounded set of real numbers, and let $B \subseteq A$ be a nonempty subset of A. Prove that B is also bounded and that $\inf(A) \leq \sup(B) \leq \sup(A)$.

(b) Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. For each $n \in \mathbb{N}$ define

$$b_n = \sup\left(\{a_m : m \in \mathbb{N} \text{ s.t. } m \geq n\}\right).$$

Prove that $\{b_n\}_{n=1}^{\infty}$ is a convergent sequence. (You may use part (a)).

Question: ________