Math 220, Sections 201/202
Final Exam
April 15, 2005
Duration: 150 minutes

Name: ______________________ Student Number: ______________________

Do not open this test until instructed to do so! This exam should have 12 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed. Turn off any call phones, pagers, etc. that could make noise during the exam. You must remain in this room until you have finished the exam.

All your solutions must be written clearly and understandably. Use complete sentences and explain why your mathematical statements are relevant to the problem. You should always write enough to demonstrate that you're not just guessing the answer. Use the backs of the pages if necessary. You will find some of the questions quite easy; try to solve these first. Good luck!

Read these UBC rules governing examinations:

(i) Each candidate must be prepared to produce, upon request, a Library/AMS card for identification.
(ii) Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
(iii) No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
(iv) Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
- Having at the place of writing any books, papers or memoranda, calculators, computers, audio or video cassette players or other memory aid devices, other than those authorized by the examiners.
- Speaking or communicating with other candidates.
- Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

(v) Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.

Problem	**Out of**	**Score**
1	6	
2	6	
3	6	
4	6	
5	6	
6	6	

Problem	**Out of**	**Score**
7	6	
8	8	
9	8	
10	8	
11	9	
Total	75	

1. **[6 pts]** The Intermediate Value Theorem states:

> For any real numbers $a < b$, and for any function $f : [a, b] \to \mathbb{R}$, if f is continuous on the interval $[a, b]$, then for every real number y between $f(a)$ and $f(b)$, there exists a real number $c \in [a, b]$ such that $f(c) = y$.

Write down the negation of this statement. (Don't worry that the negation will be a false statement—just negate the statement correctly.)

2. **[6 pts]** Consider the following two statements:

(a) For all $w \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $w < x$.

(b) There exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, $y < z$.

One of the statements is true, and the other one is false. Determine which is which. For the true one, prove that it is true; for the false one, prove that it is false.

3.

(a) **[2 pts]** Define what it means for two sets A and B to have the same cardinality.

(b) **[2 pts]** Define what it means for a set C to be countable.

(c) **[2 pts]** What does "Cantor's Diagonalization Argument" prove? (You don't have to describe the proof itself—just what it proves.)

4. **[6 pts]** Define S to be the set of all real numbers x such that *exactly one* of the two series

$$\sum_{k=1}^{\infty} \left(x - \frac{1}{2}\right)^k \quad \text{and} \quad \sum_{k=1}^{\infty} \left(x + \frac{1}{2}\right)^k$$

converges. Write down (with justification) a simple expression for S, using interval notation and set operations such as $\cup$, $\cap$, and/or $-$.

5. **[6 pts]** Let $a_1, a_2, a_3, \ldots$ be real numbers. Prove, using induction, that for all $n \in \mathbb{N}$,

$$\left| \sum_{k=1}^{n} a_k \right| \leq \sum_{k=1}^{n} |a_k|.$$

You may assume the Triangle Inequality in the form $|x + y| \leq |x| + |y|$ for all real numbers x and y.

6. **[6 pts]** Let T be the set of all natural numbers that can be written as some nonnegative number of 3's plus some nonnegative number of 5's. For example, $9 = 3 + 3 + 3$ and $10 = 5 + 5$ and $17 = 3 + 3 + 3 + 3 + 5$ are all in T, but 4 is not. Determine T (with justification).

7. **[6 pts]** Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k(k+1)}{2^k}$$

converges or diverges; prove your answer.

8. **[8 pts]** Suppose that $\{b_n\}$ is a sequence that satisfies $|b_m - b_n| < \frac{1}{m+n}$ for all $m, n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} b_n$ exists.

9. **[8 pts]** Prove, directly from the ε-definition of the limit of a sequence, that

$$\lim_{n\to\infty}\left(-\frac{2}{3}\right)^n = 0.$$

10. **[8 pts]** Suppose that $f : \mathbb{N} \to \mathbb{R}$ is a bounded function and that $\{a_n\}$ is a sequence that converges to 0. Prove that $\lim_{n\to\infty} f(n)a_n = 0$.

11. **[9 pts]** Let M be the set $M = \{x \in \mathbb{Q} \colon x^2 < 5\}$. Find, with proof, $\inf M$.