

## Math 215/255 Final Exam, December 2013

Last Name: \_\_\_\_\_ First Name: \_\_\_\_\_

Student Number: \_\_\_\_\_ Signature: \_\_\_\_\_

**Instructions.** The exam lasts 2.5 hours. **No calculators or electronic devices of any kind are permitted.** A formula sheet is attached. There are **16 pages** in this test including this cover page, blank pages, and the formula sheet. Unless otherwise indicated, show all your work.

### Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (a) speaking or communicating with other candidates, unless otherwise authorized;
  - (b) purposely exposing written papers to the view of other candidates or imaging devices;
  - (c) purposely viewing the written papers of other candidates;
  - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)-(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem #	Value	Grade
1	12	
2	12	
3	16	
4	14	
5	16	
6	18	
7	12	
Total	100	

1. Short answer questions [2 marks each; no partial credit]:

(a) Write the general solution of  $y'' + 2y' + y = 0$ .

(b) Find the *form* (but not the values of the coefficients!) of a particular solution of the nonhomogeneous ODE:  $y'' - 4y' + 5y = x \sin(x)$

(c)  $y_1(x) = x$  and  $y_2(x) = -x$  are both solutions of the first-order initial value problem:  $yy' = x$ ,  $y(0) = 0$ . Does this contradict the existence and uniqueness theorem ('Picard's theorem' or 'Cauchy-Lipshitz theorem')? Explain.

(d) Why are the Laplace transforms of 1 and  $u(t)$  (the Heaviside step function) identical?

(e) Write the second-order linear ODE  $y'' + ty' + t^2y = 0$  as a first-order linear system.

(f) Find the Laplace transform  $X(s)$  of the solution to the initial value problem:  
 $x''' + x = 0$ ,  $x(0) = 1$ ,  $x'(0) = 2$ ,  $x''(0) = 3$  (but do **not** compute  $x(t)$ ).

2. (a) [6 marks] Find the solution of

$$y' + x^3y = 3x^3, \quad y(0) = 8.$$

(b) [6 marks] Solve

$$\frac{dx}{dt} = \frac{t^2 + 1}{x^2 + 1}, \quad x(0) = -1.$$

You may leave your result in implicit form.

3. Suppose the displacement  $x(t)$  of a damped mass-spring system subject to sinusoidal forcing of amplitude  $F_0$  is modelled by:

$$x'' + 2x' + 5x = F_0 \sin(2t), \quad x(0) = 0, \quad x'(0) = 1.$$

- (a) [8 marks] Find and sketch the solution  $x(t)$  when  $F_0 = 0$  (no forcing).

- (b) [8 marks] Now take  $F_0 = 1$ , and find the steady periodic solution (the part of the solution  $x(t)$  which remains as  $t \rightarrow \infty$ ).

4. (a) [5 marks] Find the Laplace transform of the following piecewise continuous function by using either the definition of Laplace transform or the second shifting theorem involving a unit step function.

$$g(t) = \begin{cases} 0, & t < 1, \\ e^{(t-1)}, & t \geq 1. \end{cases}$$

- (b) [4 marks] Let  $Y(s) = \mathcal{L}[y]$  be the Laplace transform of the solution  $y(t)$  to the following initial value problem:

$$\begin{cases} y'' - 2y' + y = g(t), \\ y(0) = 0, \quad y'(0) = 1, \end{cases}$$

where  $g(t)$  is the function defined in (a). Determine  $Y(s)$ .

- (c) [5 marks] Calculate the inverse Laplace transform of  $Y(s)$  obtained in (b) to find the solution,  $y(t)$ , to the IVP in (b).

5. (a) [8 marks] Find the general solution  $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$  of the homogeneous linear system

$$\vec{x}' = A\vec{x}, \quad A = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix},$$

and sketch the phase portrait (i.e. some typical trajectories in the  $xy$ - plane).

- (b) [8 marks] Now find the general solution of the nonhomogenous system (with matrix  $A$  as above):

$$\vec{x}' = A\vec{x} + \begin{bmatrix} 25e^{3t} \\ 0 \end{bmatrix}.$$

*Hint: to find a particular solution, you may use either the method of variation of parameters, or the method of undetermined coefficients; for the latter, try  $\vec{x}_p(t) = \begin{bmatrix} (at + b)e^{3t} \\ (ct + d)e^{3t} \end{bmatrix}$ .*

6. Suppose the interaction between two species ( $x = x(t)$  is the “prey” while  $y = y(t)$  is the predator) can be modelled by the autonomous system:

$$\begin{cases} \frac{dx}{dt} = x(3 - x - y) \\ \frac{dy}{dt} = y(1 + x - y) \end{cases}$$

- (a) [10 marks] Determine all the critical points and classify their type and stability.
- (b) [6 marks] Sketch a phase portrait for the nonlinear system by first sketching a few trajectories near each critical point.
- (c) [2 marks] If  $x(0) = y(0) = 1$ , determine  $\lim_{t \rightarrow \infty} x(t)$  and  $\lim_{t \rightarrow \infty} y(t)$ .

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7. Consider the following initial value problem for a first-order autonomous ODE:

$$\frac{dy}{dt} = -y(y-1)^2, \quad y(0) = y_0.$$

- (a) [4 marks] Find the equilibrium solutions (critical points), determine the stability (stable or unstable) of each, and find  $\lim_{t \rightarrow \infty} y(t)$  for all possible values of the initial condition  $y_0$ .
- (b) [4 marks] If  $y_0 = 3/2$ , use Euler's method with step size  $h = 1$  to approximate the solution  $y(2)$  at time  $t = 2$ .
- (c) [4 marks] Again if  $y_0 = 3/2$ , use Euler's method with the larger step size  $h = 2$  to approximate the solution  $y(2)$  at time  $t = 2$ . Predict what will happen if you continue to use this scheme (with  $h = 2$ ) to approximate  $y(t)$  for large  $t$ . Compare this with your result from part (a), and explain any discrepancies.

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$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$	$e^{at}f(t)$	$F(s-a)$
$e^{at}$	$\frac{1}{s-a}$	$u(t-a)f(t-a)$	$e^{-as}F(s)$
$t^n$	$\frac{n!}{s^{n+1}}$	$(-t)^n f(t)$	$F^{(n)}(s)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$f'(t)$	$sF(s) - f(0)$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$
$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$	$u(t-a)$	$e^{-as}/s$
$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$	$\delta(t-a)$	$e^{-as}$
$e^{at} \sin(\omega t)$	$\frac{\omega}{(s-a)^2 + \omega^2}$	$f(t) * g(t)$	$F(s)G(s)$
$e^{at} \cos(\omega t)$	$\frac{s-a}{(s-a)^2 + \omega^2}$		

Table 1: Tables of Laplace Transforms