Be sure this examination has 3 pages, including one page that just has a table.

The University of British Columbia
Final Examinations - April 2005

Mathematics 215: *Elementary Differential Equations I*

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Closed book examination. | **Time: 2.5 hours = 150 minutes.**

Special Instructions: **No aids allowed, except for one coloured letter-size formula sheet.** Write your answers in the answer booklet(s). If you use more than one booklet, put your name and the number of booklets used on each booklet. **Show enough of your work to justify your answers.**

1. (16 points) Compute solutions for the following initial-value problems, and find the largest t-intervals in which those solutions are valid. Could those intervals change if we changed the initial value $y(2)$ to some other number $\hat{y}(2)$? Explain briefly.

(a) $t\dfrac{dy}{dt} + 3y = 6t^3, \qquad y(2) = 8.$

(b) $\dfrac{dy}{dt} + 2ty^2 = 2y^2, \qquad y(2) = 8.$

2. (16 points) Consider the differential equation

$$2y'' + 2y' + 0y = F(t),$$

where one $F(t)$ is used in part (a) below, and another is used in the rest of the question.

(a) Find all solutions in the case where $F(t) = e^{-t} + e^t + 2$.

(b) Let $F(t) = 4\cos(t)$ in the rest of this question. Then one solution is

$$y = -\cos(t) + \sin(t).$$

Check this, and find the amplitude and phase shift for that solution.

(c) What other solutions are there, for the same $F(t)$ as in part (b)?

(d) Which of those solutions have a steady state?

3. (20 points) Consider the following linear system.

$$\frac{dx_1}{dt} = -x_1 - x_2,$$

$$\frac{dx_2}{dt} = 4x_1 - x_2.$$

(a) Find the solution of this system for which $x_1(0) = 2$ and $x_2(0) = -3$. Convert any complex exponentials in your answer to a "real form" involving sines and cosines.

(b) Explain what happens to $x_1(t)$ and $x_2(t)$ and to $x_2(t)/x_1(t)$ as $t \to \infty$.

(c) Change the first equation above to $dx_1/dt = -x_1 - x_2 + 4e^{-t}$, but do *not* change the second equation or the initial conditions. Solve this new problem.

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4. (16 points) Consider the initial-value problem

$$y' = 2 + y^2 + t\,, \quad y(4) = -2.$$

(a) Use the Euler method with step size 0.1 to find an approximation to the solution at $t = 4.2$ for this problem

(b) Suppose that the error in the approximation of part (a) is about -0.175. Estimate the error that would arise if one used the same method with 10 steps of equal size to estimate $y(4.2)$, but do **NOT** carry out the method with that number of steps.

(c) Let A be the approximation to $y(4.2)$ that you found in part (a) and let B be the approximation (**not** the error) coming from 10 steps in part (b). Specify a linear combination of A and B that would be likely to give a much better approximation to $y(4.2)$.

5. (16 points) A person begins retirement with \$1,000,000 invested prudently. He/she has no other income, but the money grows at the continuous rate of 4%, except for the following. In most years, the person spends the money at the continuous rate of 7% of whatever money there is at the time. But he/she also goes on a cruise for the whole second year of retirement, and during that year the money is spent at the continuous rate of \$6,000 per month in addition to the 7% rate mentioned earlier. Let p be the value of the person's remaining money **in thousands of dollars** after t years of retirement.

(a) Explain why p satisfies the differential equation

$$\frac{dp}{dt} = \begin{cases} -0.03p - 72 & \text{when } 1 < t < 2 \\ -0.03p & \text{when } 0 < t < 1 \text{ and when } 2 < t. \end{cases}$$

(b) Solve the differential equation in part (a), for all $t \geq 0$, with $p = 1,000$ when $t = 0$.

(c) Does the person ever run out of money in this model?

6. (16 points) Find the Laplace transform $G(s)$ say, of the function g given by letting

$$g(t) = \begin{cases} 72 & \text{when } 1 < t < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then explain why the Laplace transform, $P(s)$ say, of the solution $p(t)$ in part (b) of Question 5 satisfies the condition that

$$(s + 0.03)P(s) = G(s) + 1,000.$$

Finally, write the inverse transform of $[G(s)/(s + 0.03)]$ as an integral involving the function g and another function.

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When the exam was given, this page just contained a copy of the table of Laplace transforms displayed in that chapter of the textbook.

To get a smaller pdf file, we omit that table here.

The End