

**The University of British Columbia**  
**Math 200 — Multivariable Calculus**  
**2013, December 16**

Surname: \_\_\_\_\_ First Name: \_\_\_\_\_ Student ID: \_\_\_\_\_

Section number: \_\_\_\_\_ Instructor's Name: \_\_\_\_\_

### Instructions

- **Explain** your reasoning thoroughly, and **justify** all answers (even if the question does not specifically say so).
- NO calculators or other aids are permitted.
- Duration: **2.5** hours.

Good luck, and enjoy the break.

#### Rules governing examinations

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
- Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.
- Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - (a) speaking or communicating with other candidates, unless otherwise authorized;
  - (b) purposely exposing written papers to the view of other candidates or imaging devices;
  - (c) purposely viewing the written papers of other candidates;
  - (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question	Points	Score
1	34	
2	11	
3	7	
4	7	
5	8	
6	8	
7	9	
8	8	
9	8	
Total:	100	

1. **Short Answer Problems.** Put your answer in the box provided **but show your work also**. Not all questions are of equal difficulty. Full marks will be given for correct answers placed in the box; partial credit might be given for incorrect answers. **Simplify your answers as much as possible in this question.**

6 marks

- (a) The line  $L$  has vector parametric equation  $\mathbf{r}(t) = (2 + 3t)\mathbf{i} + 4t\mathbf{j} - \mathbf{k}$ .
- Write the symmetric equations for  $L$ .

Answer:

- Let  $\alpha$  be the angle between the line  $L$  and the plane given by the equation  $x - y + 2z = 0$ . Find  $\alpha$ .

Answer:

**9 marks**

- (b) i. Find the equation of the tangent plane to the surface  $x^2z^3 + y\sin(\pi x) = -y^2$  at the point  $P = (1, 1, -1)$ .

Answer:

- ii. Let  $z$  be defined implicitly by  $x^2z^3 + y\sin(\pi x) = -y^2$ . Find  $\frac{\partial z}{\partial x}$  at the point  $P = (1, 1, -1)$ .

Answer:

- iii. Let  $z$  be the same implicit function as in part (ii), defined by the equation  $x^2z^3 + y\sin(\pi x) = -y^2$ . Let  $x = 0.97$ , and  $y = 1$ . Find the approximate value of  $z$ .

Answer:

**3 marks**

- (c) Suppose that  $u = x^2 + yz$ ,  $x = \rho r \cos(\theta)$ ,  $y = \rho r \sin(\theta)$  and  $z = \rho r$ . Find  $\frac{\partial u}{\partial r}$  at the point  $(\rho_0, r_0, \theta_0) = (2, 3, \pi/2)$ .

Answer:

**3 marks**

- (d) Let  $f(x)$  be a differentiable function, and suppose it is given that  $f'(0) = 10$ . Let  $g(s, t) = f(as - bt)$ , where  $a$  and  $b$  are constants. Evaluate  $\frac{\partial g}{\partial s}$  at the point  $(s, t) = (b, a)$ , that is, find  $\frac{\partial g}{\partial s}|_{(b,a)}$ .

Answer:

**3 marks**

- (e) Suppose it is known that the direction of the fastest increase of the function  $f(x, y)$  at the origin is given by the vector  $\langle 1, 2 \rangle$ . Find a unit vector  $\mathbf{u}$  that is tangent to the level curve of  $f(x, y)$  that passes through the origin.

Answer:

**6 marks**

- (f) Find all the points on the surface  $x^2 + 9y^2 + 4z^2 = 17$  where the tangent plane is parallel to the plane  $x - 8z = 0$ .

Answer:

**4 marks**

- (g) Find the total mass of the rectangular box  $[0, 1] \times [0, 2] \times [0, 3]$  (that is, the box defined by the inequalities  $0 \leq x \leq 1$ ,  $0 \leq y \leq 2$ ,  $0 \leq z \leq 3$ ), with density function  $h(x, y, z) = x$ .

Answer:

In the questions 2-9, justify your answers and **show all your work**. If a box is provided, write your final answer there. Unless otherwise indicated, **simplification of answers is not required in these questions**.

2. The shape of a hill is given by  $z = 1000 - 0.02x^2 - 0.01y^2$ . Assume that the  $x$ -axis is pointing East, and the  $y$ -axis is pointing North, and all distances are in metres.

3 marks

- (a) What is the direction of the steepest ascent at the point  $(0, 100, 900)$ ? (The answer should be in terms of directions of the compass).

Answer:

4 marks

- (b) What is the slope of the hill at the point  $(0, 100, 900)$  in the direction from (a)?

Answer:

4 marks

- (c) If you ride a bicycle on this hill in the direction of the steepest descent at 5 m/s, what is the rate of change of your altitude (with respect to time) as you pass through the point  $(0, 100, 900)$ ?

Answer:

- 4 marks 3. (a) Find the minimum of the function

$$f(x, y, z) = (x - 2)^2 + (y - 1)^2 + z^2$$

subject to the constraint  $x^2 + y^2 + z^2 = 1$ , using the method of Lagrange multipliers.

- 3 marks (b) Give a geometric interpretation of this problem.

5 marks
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4. (a) Find the minimum of the function  $h(x, y) = -4x - 2y + 6$  on the closed bounded domain defined by  $x^2 + y^2 \leq 1$ .

2 marks
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- (b) Explain why Question 4 gives another way of solving Question 3.

8 marks
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5. This question is about the integral

$$\int_0^1 \int_{\sqrt{3}y}^{\sqrt{4-y^2}} \ln(1+x^2+y^2) dx dy.$$

- (a) Sketch the domain of integration.
- (b) Evaluate the integral by transforming to polar coordinates.

8 marks
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6. Evaluate

$$\int_{-1}^0 \int_{-2}^{2x} e^{y^2} dy dx.$$

**9 marks**

7. Let  $a > 0$  be a fixed positive real number. Consider the solid inside both the cylinder  $x^2 + y^2 = ax$  and the sphere  $x^2 + y^2 + z^2 = a^2$ . Compute its volume.

Hint:  $\int \sin^3(\theta) d\theta = \frac{1}{12} \cos(3\theta) - \frac{3}{4} \cos(\theta) + C$ .

**8 marks**

8. (a) Sketch the surface given by the equation  $z = 1 - x^2$ .
- (b) Let  $E$  be the solid bounded by the plane  $y = 0$ , the cylinder  $z = 1 - x^2$ , and the plane  $y = z$ . Set up the integral

$$\iiint_E f(x, y, z) dV$$

as an iterated integral.

**5 marks**

9. (a) Find the volume of the solid inside the surface defined by the equation  $\rho = 8 \sin(\varphi)$  in spherical coordinates.

Hint: you do not need a sketch to answer this question; and

$$\int \sin^4(\varphi) d\varphi = \frac{1}{32}(12\varphi - 8 \sin(2\varphi) + \sin(4\varphi)) + C.$$

**3 marks**

- (b) Sketch this solid or describe what it looks like. (Hint: it is a solid of revolution).