The University of British Columbia
Final Examination - April 13, 2011
Mathematics 152
All Sections

Closed book examination. No calculators. Time: 2.5 hours

Last Name _____________________ First __________ Signature ___________

Student Number _____________________
Instructor: _____________________

Special Instructions:
No books, notes, or calculators are allowed. Show all your work, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBC card for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practises shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  - Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  - Speaking or communicating with other candidates.
  - Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.
Part A - Short Answer Questions, 1 mark each

For questions A1-A3 below, \( \mathbf{x} \) and \( \mathbf{y} \) are the vectors given below:
\[
\mathbf{x} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix} \\
\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

A1: Compute \( \mathbf{x} + \mathbf{y} \).

A2: Compute \( \mathbf{x} \times \mathbf{y} \) (the cross product of \( \mathbf{x} \) and \( \mathbf{y} \)).

A3: Compute the projection of \( \mathbf{x} \) onto the direction of \( \mathbf{y} \).

A4: Circle the one correct answer below. A linear system of two equations in three unknowns has
(a) always a unique solution.
(b) either a unique solution or no solutions.
(c) either a unique solution or an infinite number of solutions.
(d) either no solutions or an infinite number of solutions.
(e) either a unique solution or exactly two distinct solutions.

A5: Find the inverse of the matrix
\[
\begin{bmatrix}
2 & 3 \\
1 & 2
\end{bmatrix}
\]

A6: What is output after the following lines of MATLAB code?
\[
\begin{align*}
\text{A} &= \begin{bmatrix} 1 & 2 & 3; & 4 & 5 & 6; & 7 & 8 & 9 \end{bmatrix}; \\
\text{A}(2,:) &= 
\end{align*}
\]

A7: Write the matrix \( \mathbf{A} \) that would result from the following lines of MATLAB code:
\[
\begin{align*}
\text{A} &= \text{zeros}(3,2); \\
\text{A}(1,1) &= 5; \\
\text{A}(2,2) &= 3; \\
\text{A}(3,1) &= 1;
\end{align*}
\]
A8: Write the matrix $A$ that would result from the following lines of MATLAB code:

```matlab
A = zeros(4,4);
for k=1:4
    A(k,k) = -k;
end
for k=1:4
    A(1,k) = 1;
end
```

For questions **A9-A11** below consider the circuit in the diagram below:

![Circuit Diagram](image)

**A9:** List the unknowns in the linear system for this circuit using the technique of loop currents you learned in the computer labs this term.

**A10:** In terms of these unknowns, write the linear equation that represents Kirchhoff’s law of voltage drops around the third loop (corresponding to $i_3$) in the circuit above.

**A11:** Write a linear equation that expresses the current through the current source 1 in terms of the loop currents in the diagram.
A12. Write the matrix of the rotation which transforms \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) to \( \begin{bmatrix} -1 \\ 0 \end{bmatrix} \).

The unit square in \( \mathbb{R}^2 \) is the set of all points \((x, y)\) with \(0 \leq x \leq 1\) and \(0 \leq y \leq 1\). In questions A13 and A14 below, suppose each point in the unit square undergoes a linear transformation that has the given matrix representation \( A \). Draw a labelled sketch of the transformed points.

A13: \( A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \)

A14: \( A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \)

A15: Consider the linear transformation \( T \) which takes vectors in \( \mathbb{R}^2 \), first rotates them by \( \pi/6 \) \((30^\circ)\) counter-clockwise and then projects the result onto the direction \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

Find \( T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \). Note: \( \cos \pi/6 = \sqrt{3}/2 \) and \( \sin \pi/6 = 1/2 \). Your final answer should not contain trigonometric functions.

A16: Find a vector of length 1 that is parallel to the line given by the equations

\[
\begin{align*}
y - z &= 1 \\
2x + y + z &= 2
\end{align*}
\]
For questions **A17-A19** below, \( u \) and \( z \) are the complex numbers given below:

\[
\begin{align*}
uc &= 4 - 2i \\
z &= -1 + 3i
\end{align*}
\]

**A17**: Compute \( uz \). Your answer should be in the form \( a + ib \) where \( a \) and \( b \) are real numbers.

**A18**: Compute \( |z| \).

**A19**: Evaluate \( u/z \). Put your answer in the form \( a + ib \) where \( a \) and \( b \) are real numbers.

**A20**: Write \(-1 + i\) in polar form \( re^{i\theta} \).

**A21**: Suppose that \( A \) is a \( 12 \times 12 \) invertible matrix and \( b \) is a column vector with 12 entries. Both \( A \) and \( b \) have been entered into MATLAB. What commands would you use to compute the vector \( x \) that solves the linear system

\[
A x = b
\]
Questions A22-A24 below concern a random walk with three states that has transition matrix

\[
P = \begin{bmatrix}
  8/10 & 3/10 & 1/10 \\
  2/10 & 3/10 & 7/10 \\
  0 & 4/10 & 2/10 
\end{bmatrix}.
\]

A22: If the walker is equally likely to start in each of the three states, what is the chance that she will be in state 3 at the next time step?

A23: If the walker starts in state 2, which state will she most likely be in after 2 time steps?

A24: What is the equilibrium probability vector for the random walk?
A25: The matrix

\[ A = \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \]

contains a real parameter \( p \). For what values of \( p \) does \( A \) have three real, distinct eigenvalues?

A26: For which values of the parameters \( a, b \) and \( c \) is the matrix

\[ B = \begin{bmatrix} a & a & a \\ a & a & b \\ a & b & c \end{bmatrix} \]

invertible?
A27: Circle all of the following statements that are true for all \( n \times n \) invertible matrices \( A \):

(a) \( A \) has rank \( n \).
(b) The reduced row echelon form of \( A \) is the identity matrix.
(c) The homogeneous problem \( Ax = 0 \) has only the trivial solution \( x = 0 \).
(d) \( A \) does not have \( \lambda = 0 \) as an eigenvalue.
(e) \( \det(A) \neq 0 \)

A28: Let

- \( \text{Rot}_\alpha \) be the matrix for 2D rotation counterclockwise with angle \( \alpha \)
- \( \text{Refl}_\beta \) 2D reflection through the line that makes angle \( \beta \) with the positive \( x \) axis
- \( \text{Proj}_\theta \) 2D projection onto the line that makes angle \( \theta \) with the positive \( x \) axis.

Circle all the matrix products below that are the identity matrix for all \( \alpha \), \( \beta \), and \( \theta \).

(a) \( \text{Refl}_\beta \text{Refl}_\beta \)
(b) \( \text{Rot}_\alpha \text{Rot}_{-\alpha} \)
(c) \( \text{Proj}_\theta \text{Proj}_\theta \)
(d) \( \text{Proj}_\theta \text{Proj}_{\pi/2-\theta} \)
(e) \( \text{Rot}_\alpha \text{Refl}_{\alpha/2} \)

For questions A29 and A30 below, decide if the statements are true or false. In each case, give a brief justification of your conclusion.

A29: For all \( n \times n \) matrices \( A \) and \( B \) such that \( \det(A) = \det(B) = 0 \), \( \det(A + B) = 0 \) also.

A30: If vectors \( \mathbf{v} \) and \( \mathbf{w} \) in \( \mathbb{R}^3 \) are linearly independent and lie in the plane \( P \) that goes through the origin then \( \mathbf{v} \times (\mathbf{v} \times \mathbf{w}) \) also lies in \( P \).
Part B - Long Answer Questions, 5 marks each

B1: Three identical fields of size one hectare are planted with different amounts of three kinds of wheat (types 1, 2 and 3). The first field is divided in three equal parts with one type of wheat planted on each part. It yields 12 tons of total harvest. The second field is divided in two equal parts with type 2 planted on one part and type 3 planted on the other. It yields 10 tons. The third field is divided in two equal parts with type 1 planted on one part and type 2 on the other. It yields 16 tons.

(a) [2 marks] Let

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]

be the vector of unknowns, where \( x_k \) is the yield in tons per hectare of wheat type \( k \). Describe the information above as a linear system in the form

\[ Ax = b \]

(write \( A \) and \( b \) with specific values).

(b) [1] Write the system you found above in augmented matrix form.

(c) [2] Solve the system above using Gaussian elimination on the augmented matrix. What is the yield in tons per hectare for each type of wheat?
B2: Consider the differential equation system

\[ y' = Ay \]

where \( A \) has eigenvalues \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \) with corresponding eigenvectors

\[ k_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad k_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

(a) [2 marks] What is the general solution of the system of differential equations?

(b) [3] Find the solution that matches the initial conditions \( y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).
B3: Let \( A \) be the \( 2 \times 2 \) matrix which transforms \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) to \( \begin{bmatrix} 2 \\ 2 \end{bmatrix} \) and transforms \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) to \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

(a) [2 marks] What are the eigenvalues and eigenvectors of \( A \)?

(b) [1] Is \( A \) invertible? Justify briefly.

(c) [2] What is the matrix \( A \)?
B4: Let $a = [1, 2, -1]$. Consider the transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x) = x \times a$ for all $x$ in $\mathbb{R}^3$, where $\times$ is the cross product in $\mathbb{R}^3$.

(a) [1 mark] Show that $T$ is a linear transformation.

(b) [2] Find the matrix $A$ that corresponds to $T$.

(c) [1] Compute $\text{det} A$.

(d) [1] Find any nonzero vector $x$ such that $T(x) = 0$. 
B5: Consider the matrix below:

\[ A = \begin{bmatrix} 3 & 1 & -3 \\ -1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}. \]

It is known that \( \lambda_1 = 1 \) is an eigenvalue of \( A \).

(a) [2 marks] Find an eigenvector associated with the eigenvalue \( \lambda_1 = 1 \).

(b) [2] Find the two other eigenvalues \( \lambda_2 \) and \( \lambda_3 \) of \( A \).

(c) [1] Find eigenvectors associated with \( \lambda_2 \) and \( \lambda_3 \).
B6: Consider the linear systems described below. In cases (a)-(d), write a single, possible reduced row echelon form of the augmented matrix of the system.

(a) [1 mark] Two equations in two unknowns with unique solution \[
\begin{bmatrix}
2 \\
3
\end{bmatrix}.
\]

(b) [1] Two equations in three unknowns with solutions
\[
\begin{bmatrix}
1 \\
2 \\
0
\end{bmatrix} + t
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}
\]
where \( t \) is a parameter.

(c) [1] Three equations in three unknowns with no solution.

(d) [1] Three equations in two unknowns with no solution.

(e) [1] Repeat part (d) but in this case only, list all possible corresponding reduced row echelon forms.

The End