

The University of British Columbia

Final Examination - April 23, 2010

Mathematics 152

All Sections

Closed book examination. No calculators.

Time: 2.5 hours

Last Name \_\_\_\_\_ First \_\_\_\_\_ Signature \_\_\_\_\_

Student Number \_\_\_\_\_

Section : \_\_\_\_\_

Instructor : \_\_\_\_\_

Special Instructions:

No books, notes, or calculators are allowed. Show all your work, little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCCard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practises shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  - (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  - (b) Speaking or communicating with other candidates.
  - (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

part A		30
B1		5
B2		5
B3		5
B4		5
B5		5
B6		5
Total		60

**Part A - Short Answer Questions, 1 mark each**

For questions **A1-A3** below,  $\mathbf{x}$  and  $\mathbf{y}$  are the vectors given below:

$$\mathbf{x} = [1, -1, -1]$$

$$\mathbf{y} = [1, 1, 2]$$

**A1:** Compute  $\|\mathbf{x}\|$ .

**A2:** Compute  $\mathbf{x} \times \mathbf{y}$ .

**A3:** Are  $\mathbf{x}$  and  $\mathbf{y}$  linearly independent? Justify briefly.

**A4:** Circle the *one* correct answer below. A linear system of four equations in three unknowns has

- (a) always a unique solution.
- (b) either a unique solution or no solutions.
- (c) either a unique solution or an infinite number of solutions.
- (d) either no solutions or an infinite number of solutions.
- (e) either no solutions, a unique solution or an infinite number of solutions.

**A5:** If 2 and 3 are both eigenvalues of a  $2 \times 2$  matrix  $A$  then  $A$  must be invertible. Justify this statement briefly.

**A6:** For what values of the parameter  $a$  does the matrix

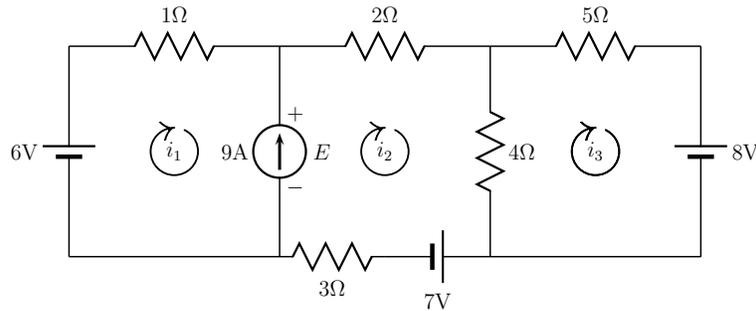
$$\begin{bmatrix} 1 & 0 & a \\ -1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

*not* have an inverse?

**A7:** Find the inverse of the matrix  $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

**A8:** What is  $\cos \theta$  where  $\theta$  is the angle between the vectors  $[1, 2, 3]$  and  $[-1, 5, -2]$ ?

For questions **A9-A11** below consider the circuit in the diagram below:



**A9:** List the unknowns in the linear system for this circuit set up using the techniques you learned in the computer labs this term.

**A10:** In terms of these unknowns, write the linear equation that represents Kirchhoff's voltage law around the second (middle) loop in the circuit above.

**A11:** Write a linear equation that expresses the current through the current source in terms of the loop currents in the diagram.

**A12:** Write the matrix **A** that would result from the following lines of MATLAB code:

```
A = zeros(2,3);
A(1,1) = 2;
A(2,3) = 3;
```

**A13.** Write the matrix **A** that would result from the following lines of MATLAB code:

```
A = zeros(3,3);
for i=1:2
    A(i,i) = -5;
    A(i,i+1) = 1;
end
A(3,3) = -2;
```

**A14:** What is output after the following lines of MATLAB code?

```
A = [1 2 3; 4 5 6];
A(:,1)
```

**A15:** List all eigenvalues of  $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

**A16:** Write the matrix  $A$  so that the equation

$$s \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$$

can be represented as a matrix equation  $A \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$ .

**A17:** The matrix

$$P = \begin{bmatrix} 4/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}$$

represents projection onto a line in 2D. Write this line in the form  $y = mx$  with  $m$  determined.

**A18:** Find the intersection of the line  $L$  and the plane  $P$  given below

$$L : \quad \mathbf{x} = (1, 1, 0) + t(1, 0, -2)$$

$$P : \quad x + y + z = 2$$

**A19:** If  $R$  is the 2D counter-clockwise rotation by  $\pi/4$  and  $Q$  is the 2D reflection through the line  $y = x$ , is  $QR = RQ$ ? Justify briefly.

For questions **A20-21** below,  $T$  is a linear transformation from 2D to 3D such that

$$T \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad T \left( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$$

**A20:** What is  $T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$ ?

**A21:** What is the matrix representation of  $T$ ?

For questions **A22-A24** below,  $u$  and  $z$  are the complex numbers given below:

$$u = 3 + 2i$$

$$z = 1 - i$$

**A22:** Evaluate  $2u - z$ . Your answer should be in the form  $a + ib$  where  $a$  and  $b$  are real numbers.

**A23:** Evaluate  $\bar{u}$

**A24:** Evaluate  $u/z$ . Put your answer in the form  $a + ib$  where  $a$  and  $b$  are real numbers.

**A25:** Evaluate  $(1 + i)^{10}$  in the form  $a + ib$ . *Hint:* Write  $1 + i$  in polar form.

**A26:** Suppose  $P$  was the transition matrix for a random walk with 12 states that had been entered into MATLAB. What commands would you use to compute the probability that if the system started in state 1, it would be in state 5 after 10 time steps?

**A27:** Suppose that  $A$  is an  $8 \times 8$  invertible matrix and  $\mathbf{b}$  is a column vector with 8 entries. Both  $A$  and  $\mathbf{b}$  have been entered into MATLAB. What commands would you use to compute the vector  $\mathbf{x}$  that solves the linear system

$$A\mathbf{x} = \mathbf{b}?$$

**A28:** Circle *all* of the following statements that are true for all invertible matrices  $A$ , and all matrices  $B$  and  $C$  of the same size as  $A$  and all scalars  $r \neq 0$ :

(a)  $(AB)C = A(BC)$

(b)  $(A + B)(A - B) = A^2 - B^2$

(c)  $A^T B^T = (AB)^T$

(d)  $(A^{-1})^T = (A^T)^{-1}$

(e)  $(rA)^{-1} = \frac{1}{r}A^{-1}$

**A29:** Circle *all* statements below that are true for any vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  in  $R^3$ .

(a)  $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

(b)  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are linearly independent

(c)  $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

(d)  $\mathbf{a} \cdot \mathbf{b} = 0$

(e)  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$  is not defined

**A30:** Find the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 3 & 2 \\ 1 & 1 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}$$

*Hint:* do row operations to simplify the computation.

**Part B - Long Answer Questions, 5 marks each**

**B1:** Three friends, Hartosh, Mark and Keiko decide to paint houses over the summer. Hartosh paints twice as fast as Mark. Hartosh and Keiko paint a home with six rooms in 8 hours. All three together paint a home with 14 rooms in 16 hours.

- (a) [2 marks] Let  $\mathbf{x} = (x_1, x_2, x_3)^T$  be the vector of unknowns, where  $x_1$  is the number of rooms that Hartosh can paint in an hour, and  $x_2$  and  $x_3$  are respectively the number of rooms that Mark and Keiko can paint in an hour. Describe the information above as a linear system in the form

$$A\mathbf{x} = \mathbf{b}$$

(write  $A$  and  $\mathbf{b}$  with specific values).

- (b) [1] Write the system you found above in augmented matrix form.
- (c) [2] Solve the system above using Gaussian elimination on the augmented matrix. How many room can each of the three friends paint in an hour?

**B2:** The matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

is known to have eigenvalues 1 and 4.

- (a) [2 marks] Find the eigenvector of  $A$  that corresponds to eigenvalue 4.
- (b) [3] Find two linearly independent eigenvectors that correspond to eigenvalue 1.

**B3:** A random walk problem with two states has a transition matrix

$$P = \begin{bmatrix} 1/3 & 1/4 \\ 2/3 & 3/4 \end{bmatrix}.$$

- (a) [1 mark] What is the probability that when the walker is in state 1 it moves to state 2?
- (b) [2] The eigenvalues of  $P$  are 1 and  $1/12$ . Find corresponding eigenvectors.
- (c) [1] Assume that the walker starts in state 1. Give an explicit formula that only depends on  $n$  for the probability it is in state 1 after  $n$  time steps.
- (d) [1] After many time steps what state is the walker more likely to be? Justify briefly.

**B4:** Below are listed augmented matrices for linear systems that have been put in reduced row echelon form with row operations. In each case write one of the following:

- the solution if it is unique
- all solutions if there are more than one
- state that there are no solutions

(a) [1 mark]  $\left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

(b) [1]  $\left[ \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

(c) [1]  $\left[ \begin{array}{cccc|c} 1 & 0 & 4 & 5 & 1 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$

(d) [1]  $\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right]$

(e) [1]  $\left[ \begin{array}{cccc|c} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 2 & -2 & 2 \end{array} \right]$

**B5:** Consider the differential equation system

$$\mathbf{y}' = A\mathbf{y}$$

where  $A$  has an eigenvalue  $\lambda_1 = 1 + 2i$  with corresponding eigenvector

$$\mathbf{k}_1 = \begin{bmatrix} 2 + 3i \\ 1 \end{bmatrix}$$

*Recall* that for real matrices, eigenvalues and eigenvectors come in complex conjugate pairs.

- (a) [2 marks] Write the general solution of the system of differential equations. This can be written in real or complex form.
- (b) [3] Find the real form (no imaginary terms) of the solution that satisfies initial conditions  $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

**B6:** Consider the triangle  $T$  in three dimensions with vertices  $(0, 1, 2)$ ,  $(1, 1, 5)$  and  $(-1, 2, 2)$ . Consider also the plane  $P$  that contains  $T$ .

(a) [1 point] What is the area of  $T$ ?

(b) [1] What is the normal (perpendicular) direction to  $P$ ?

(c) [1] Write an equation for  $P$  in the form

$$x + by + cz = d$$

with  $b$ ,  $c$  and  $d$  determined.

(d) [1] Verify that  $(0, 0, -1)$  is on the plane  $P$ .

(e) [1] Is  $(0, 0, -1)$  in the triangle  $T$ ? Justify.