Instructions

This exam consists of 7 questions worth as noted for a total of 100.
Duration: 2\frac{1}{2} hours.
Show all work and calculations and explain your reasoning thoroughly.
Read all questions; They are not in order of difficulty.
No calculators or other aids are permitted.
Make sure this exam has 11 pages including this cover page.

Good luck, and happy holydays.

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1. **Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks. Not all questions are of equal difficulty. Full marks will be given for correct answers placed in the box; at most 1 mark will be given for incorrect answers. Unless otherwise stated, it is not necessary to simplify your answers in this question.

(a) Evaluate $f'(x)$ if $f = e^{x^2 + \cos x}$.

(b) Evaluate $\lim_{x \to 1} \frac{x^3 - e^{x-1}}{\sin(\pi x)}$.

(c) Find $f(2)$ if $f'(x) = \pi f(x)$ for all $x$, and $f(0) = 2$. 

Answer
(d) Evaluate $f'(x)$ if $f(x) = \sqrt{\frac{x-1}{x+1}}$.

(e) Find $\frac{dy}{dx}$ if $xy + e^x + e^y = 1$.

(f) Use a linear approximation to estimate $\tan^{-1}(1.1)$, using $\tan^{-1} 1 = \pi/4$.

(g) Let $f(x) = x + \cos x$, and $g(y) = f^{-1}(y)$ be the inverse function. Determine $g'(y)$. 

Answer
(h) Find constants $a, b$ so that the following function is differentiable:

$$f(x) = \begin{cases} \frac{ax^2 + b}{x + 1}, & x \leq 1 \\ e^x, & x > 1. \end{cases}$$

Answer

(i) Find $\lim_{n \to \infty} \frac{(n+1)^4 \sin n}{n^6 + \sin n}$.

Answer

(j) Perform one iteration of Newton’s method for finding a root of $x - \cos x$, starting with $x_0 = 0$.

Answer
(k) For which values of $a$ is the function $f(x) = \begin{cases} 0 & x \leq 0 \\ x^a \sin\left(\frac{1}{x}\right) & x > 0 \end{cases}$ differentiable at 0?

Answer

(l) Find $c$ so that $\lim_{x \to 0} \frac{1 + cx - \cos x}{e^x - 1}$ exists.

Answer

(m) Find the point promised by the mean value theorem for the function $e^x$ on the interval $[0, T]$.

Answer

(n) Find the limit of $a_n = \sqrt{n^2 + 5n} - n$.

Answer
Long answer questions. For questions 2–7, give detailed and justified answers.

(10 points) 2. Find the maximal possible volume of a cylinder with surface area $A$. (The surface area consists of two discs and a $(2\pi r \times h)$ strip.)
(10 points)  3.  (a) Use the definition of the limit to prove \(\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}\).

(b) Use the formal definition of the derivative to compute \(f'(x)\) if \(f(x) = \sqrt{1 + x}\). (You do not need to use the formal definition of limit.)
4. Consider the function \( f(x) = xe^{-x^2/2} \).

(a) Find the limit of \( f \) as \( x \to \infty \) and \( x \to -\infty \).
(b) Find inflection points, intervals of increase, decrease, convexity and concavity. You may use without proof the formula \( f''(x) = (x^3 - 3x)e^{-x^2/2} \).
(c) Find local and global minima and maxima.
(d) Use all the above to draw a graph for \( f \). Indicate all special points on the graph.
5. Suppose $f(0) = 0$ and $f'(x) = \frac{1}{1 + e^{-f(x)}}$. Prove that $f(100) < 100$. 

(8 points)
6. (10 points) (a) Find $P_8$: the Taylor polynomial of order 8 for $f(x) = e^{x^2}$ around $x = 0$.
   (b) Use this to find $f^{(8)}(0)$.
   (c) Give an upper bound on the error $|f(x) - P_8(x)|$ that is valid for all $x \in [-1, 1]$. 
(8 points) 7. (a) If in a sequence \((a_0, a_1, \ldots)\), each term is the average of the previous two, show that the sequence converges.

(4 bonus) (b) Find the limit in terms of \(a_0\) and \(a_1\).