Math 120 Final Exam    December 2006    2.5 hours.

There are 11 pages in this test including this cover page. No calculators, books, notes, or electronic devices of any kind are permitted. Unless otherwise indicated, show all your work.

Rules governing formal examinations:

1. Each candidate must be prepared to produce, upon request, a Library/AMS card for identification;
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action;
   (a)  Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   (b)  Speaking or communicating with other candidates;
   (c)  Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received;
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator; and
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

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I have read and understood the instructions and agree to abide by them.

Signed: ________________________________
1. ([42 marks]) **Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks. Full marks will be given for correct answers placed in the box, but at most 1 mark will be given for incorrect answers. Unless otherwise stated, it is not necessary to simplify your answers in this question.

(a) Evaluate \( \lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 16} \).

(b) Evaluate \( f'(2) \) if \( f(x) = g(x/h(x)) \), \( h(2) = 2 \), \( h'(2) = 3 \), \( g'(1) = 4 \).

(c) Find the value of the constant \( a \) for which \( \lim_{x \to -2} \frac{x^2 + ax + 3}{x^2 + x - 2} \) exists.
(d) Find the values of the constants $a$ and $b$ for which

$$f(x) = \begin{cases} 
  x^2 & x \leq 2 \\
  ax + b & x > 2 
\end{cases}$$

is differentiable everywhere.

(e) Find the derivative of $e^x \cos(x)$.

(f) If $x^2y^2 + x\sin(y) = 4$, find $dy/dx$.

(g) The mass of a sample of Polonium-210, initially 6 g., decreases at a rate proportional to the mass. After one year, 1 g. gram remains. What is the half-life (the time it takes for the sample to decay to half its original mass)?
(h) Find the \((x, y)\) coordinates of all points where the graph of the parametric curve \(x = \cos(t^3), y = \sin(t^3)\) has a horizontal tangent.

(i) Find the derivative of \((\tan(x))^x\).

(j) Using a suitable linear approximation, estimate \((8.06)^{2/3}\). Give your answer as a fraction in which both the numerator and denominator are integers.

(k) \(f(x) = e^x + x\) is one-to-one. Find \((f^{-1})'(e+1)\).
(l) Find the rate of change of the area of the annulus \( \{ (x, y) \mid r^2 \leq x^2 + y^2 \leq R^2 \} \)
(i.e. the points inside the circle of radius \( R \) but outside the circle of radius \( r \)) if \( R = 3, r = 1, \frac{dR}{dt} = 2, \) and \( \frac{dr}{dt} = 7. \)

(m) The function \( f(x) = x^2 - 1 \) has roots at \( x = -1 \) and \( x = 1. \) Find an initial guess \( x_0 \) for Newton’s method so that the next approximation \( x_1 \) is larger than 100.

(n) For the function
\[
f(x) = \begin{cases} 
0 & x \leq 0 \\
\frac{\sin(x)}{\sqrt{x}} & x > 0
\end{cases},
\]
write in the box the (roman) number of the correct statement from the list:

i. \( f \) is undefined at \( x = 0 \)
ii. \( f \) is neither continuous nor differentiable at \( x = 0 \)
iii. \( f \) is continuous but not differentiable at \( x = 0 \)
iv. \( f \) is differentiable but not continuous at \( x = 0 \)
v. \( f \) is both continuous and differentiable at \( x = 0 \)
Full-Solution Problems. In questions 2-7, justify your answers and show all your work.

2. ([11 marks]) A rectangle is inscribed in a semicircle of radius $R$ so that one side of the rectangle lies along a diameter of the semicircle. Find the largest possible perimeter of such a rectangle, if it exists, or explain why it does not. Do the same for the smallest possible perimeter.
3. ([16 marks]) The function $f(x)$ is defined by

$$f(x) = \begin{cases} 
    e^x & \text{if } x < 0 \\
    \frac{x^2 + 3}{3(x+1)} & \text{if } x \geq 0
\end{cases}$$

(a) Explain why $f(x)$ is continuous everywhere.

(b) Determine all of the following if they are present:

i. $x$-coordinates of local maxima and minima, intervals where $f(x)$ is increasing or decreasing;

ii. intervals where $f(x)$ is concave upwards or downwards;

... question continued on next page
iii. equations of any asymptotes (horizontal, vertical, or slant).

(c) Sketch the graph of \( y = f(x) \), giving the \((x, y)\) coordinates for all points of interest above.
4. ([6 marks]) There are two distinct straight lines that pass through the point $(1, -3)$ and are tangent to the curve $y = x^2$. Find equations for these two lines.

5. ([4 marks]) Evaluate

$$\lim_{x \to 0} x^{1/101} \sin(x^{-100})$$

or explain why this limit does not exist. Give a complete justification of your answer.
6. ([11 marks])

(a) Find the third-order Taylor polynomial for \((1 - 3x)^{-1/3}\) around \(x = 0\).

(b) Evaluate

\[
\lim_{x \to 0} \frac{\sin(x)e^{2x} + 1 - (1 - 3x)^{-1/3}}{x^3}.
\]
7. ([10 marks])

(a) State, in terms of a limit, what it means for \( f(x) = x^3 \) to be differentiable at \( x = 0 \).

(b) Use the definition of limit to prove that \( x^3 \) is differentiable at \( x = 0 \).