Name: ______________________ Student Number: ______________________

# Math 120 Final Exam December 2006 2.5 hours.

There are **11 pages** in this test including this cover page. **No calculators, books, notes, or electronic devices of any kind are permitted. Unless otherwise indicated, show all your work.**

Rules governing formal examinations:

1. Each candidate must be prepared to produce, upon request, a Library/AMS card for identification;
2. Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions;
3. No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination;
4. Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action;
   (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
   (b) Speaking or communicating with other candidates;
   (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received;
5. Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator; and
6. Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

| Problem # | Value | Grade |
|---|---|---|
| 1 | 42 | |
| 2 | 11 | |
| 3 | 16 | |
| 4 | 6 | |
| 5 | 4 | |
| 6 | 11 | |
| 7 | 10 | |
| Total | 100 | |

**I have read and understood the instructions and agree to abide by them.**

Signed: ______________________

1. ([42 marks]) **Short-Answer Questions.** Put your answer in the box provided but show your work also. Each question is worth 3 marks. Full marks will be given for correct answers placed in the box, but at most 1 mark will be given for incorrect answers. Unless otherwise stated, it is not necessary to simplify your answers in this question.

   (a) Evaluate $\lim_{x\to 4} \frac{x^2-4x}{x^2-16}$.

   (b) Evaluate $f'(2)$ if $f(x) = g(x/h(x))$, $h(2) = 2$, $h'(2) = 3$, $g'(1) = 4$.

   (c) Find the value of the constant $a$ for which $\lim_{x\to -2} \frac{x^2+ax+3}{x^2+x-2}$ exists.

(d) Find the values of the constants $a$ and $b$ for which

$$f(x) = \begin{cases} x^2 & x \leq 2 \\ ax + b & x > 2 \end{cases}$$

is differentiable everywhere.

(e) Find the derivative of $e^{x\cos(x)}$.

(f) If $x^2y^2 + x\sin(y) = 4$, find $dy/dx$.

(g) The mass of a sample of Polonium-210, initially 6 $g$., decreases at a rate proportional to the mass. After one year, 1 $g$. gram remains. What is the half-life (the time it takes for the sample to decay to half its original mass)?

(h) Find the $(x, y)$ coordinates of all points where the graph of the parametric curve $x = \cos(t^3)$, $y = \sin(t^3)$ has a horizontal tangent.

(i) Find the derivative of $(\tan(x))^x$.

(j) Using a suitable linear approximation, estimate $(8.06)^{2/3}$. Give your answer as a fraction in which both the numerator and denominator are integers.

(k) $f(x) = e^x + x$ is one-to-one. Find $(f^{-1})'(e+1)$.

(l) Find the rate of change of the area of the annulus $\{ (x, y) \mid r^2 \leq x^2 + y^2 \leq R^2 \}$ (i.e. the points inside the circle of radius $R$ but outside the circle of radius $r$) if

$R = 3$, $r = 1$, $dR/dt = 2$, and $dr/dt = 7$.

(m) The function $f(x) = x^2 - 1$ has roots at $x = -1$ and $x = 1$. Find an initial guess $x_0$ for Newton's method so that the next approximation $x_1$ is larger than 100.

(n) For the function

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{\sin(x)}{\sqrt{x}} & x > 0 \end{cases},$$

write in the box the (roman) number of the correct statement from the list:

i. $f$ is undefined at $x = 0$
ii. $f$ is neither continuous nor differentiable at $x = 0$
iii. $f$ is continuous but not differentiable at $x = 0$
iv. $f$ is differentiable but not continuous at $x = 0$
v. $f$ is both continuous and differentiable at $x = 0$

**Full-Solution Problems.** In questions 2-7, justify your answers and **show all your work**.

2. ([11 marks]) A rectangle is inscribed in a semicircle of radius $R$ so that one side of the rectangle lies along a diameter of the semicircle. Find the largest possible perimeter of such a rectangle, if it exists, or explain why it does not. Do the same for the smallest possible perimeter.

3. ([16 marks]) The function $f(x)$ is defined by

$$f(x) = \begin{cases} e^x & x < 0 \\ \frac{x^2+3}{3(x+1)} & x \geq 0 \end{cases}$$

(a) Explain why $f(x)$ is continuous everywhere.

(b) Determine all of the following if they are present:

i. $x$-coordinates of local maxima and minima, intervals where $f(x)$ is increasing or decreasing;

ii. intervals where $f(x)$ is concave upwards or downwards;

· · · question continued on next page

iii. equations of any asymptotes (horizontal, vertical, or slant).

(c) Sketch the graph of $y = f(x)$, giving the $(x, y)$ coordinates for all points of interest above.

4. ([6 marks]) There are two distinct straight lines that pass through the point $(1, -3)$ and are tangent to the curve $y = x^2$. Find equations for these two lines.

5. ([4 marks]) Evaluate

$$\lim_{x \to 0} x^{1/101} \sin(x^{-100})$$

or explain why this limit does not exist. Give a complete justification of your answer.

6. ([11 marks])

(a) Find the third-order Taylor polynomial for $(1-3x)^{-1/3}$ around $x=0$.

(b) Evaluate

$$\lim_{x\to 0}\frac{\sin(x)e^{2x}+1-(1-3x)^{-1/3}}{x^3}.$$

7. ([10 marks])

(a) State, in terms of a limit, what it means for $f(x) = x^3$ to be differentiable at $x = 0$.

(b) Use the definition of limit to prove that $x^3$ is differentiable at $x = 0$.