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The University of British Columbia
Sessional Examinations - April 2013

Mathematics 180

Differential Calculus with Applications to Physical Sciences and Engineering

Closed book examination

Time: 2.5 hours

Last Name: _____ First Name: _____

Student Number: _____ Instructor's Name: _____

Signature: _____ Section Number: _____

READ AND OBSERVE THE FOLLOWING EXAM RULES

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) — (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1		9
2		9
3		12
4		12
5		8
6		14
7		10
8		10
9		10
10		6
Total		100

Marks

Short-Answer Questions. Questions 1 – 4 are short-answer questions. Put your answers in the boxes provided. **Simplify your answers as much as possible, and show your work.** Each question is worth 3 marks, but not all questions are of equal difficulty. Full marks will be given for a correct, simplified answer placed in the answer box.

- [9] 1. (a) Evaluate the following limit:

$$\lim_{x \rightarrow -1} \frac{x^2 + 3x + 2}{x^2 - x - 2}$$

Answer

- (b) Evaluate the following limit, or determine that it does not exist: $\lim_{x \rightarrow 1^-} \frac{1 - |x|}{|1 - x|}$

Answer

- (c) Find the *second* derivative of $f(x) = e^{\sin 2x}$.

Answer

- [9] 2. (a) Calculate the derivative of $g(s) = \tan^{-1}(\ln s)$. *Note:* arctan is an alternate notation for \tan^{-1} .

Answer

- (b) Find a function $f(x)$ that satisfies $f'(x) = x^2 + e^x$ and $f(0) = 0$.

Answer

- (c) Determine the equation of the slant asymptote of $f(x) = \frac{3x^3 + 2x^2 - 1}{x^2 - 4x + 3}$

Answer

- [12] 3. (a) Determine an equation of the tangent line at $(1, 5)$ to the curve $y = x^3 + 4$.

Answer

- (b) Find an equation of the tangent line at $(1, 1)$ to the curve $x \sin(xy - y^2) = x^2 - 1$.

Answer

- (c) A bead is moving on a straight wire so that its position x given as a function of time t is $x(t) = t^3 + t^2 + 1$. Find the value $a > 0$ so that the velocity of the bead at time a is equal to 16.

Answer

- (d) Suppose the function f is defined as

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ Ax^2 + B & \text{if } x \geq 0 \end{cases}$$

for constants A and B . For what value or values of A and B is f differentiable for all x ?

Answer

- [12] 4. (a) If $f(1) = 2$ and $f'(1) = 3$ use a linear approximation to estimate the value $f(1.15)$.

Answer

- (b) In the previous question, if it is known that $|f''(x)| < 2$ for all x , what is the maximum absolute value of the error in your approximation?

Answer

- (c) Evaluate $\lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{\cos(x) - 1}$

Answer

- (d) Evaluate $\lim_{x \rightarrow \infty} \left(1 - \frac{2}{x}\right)^x$

Answer

Full-Solution Problems. In questions 5–10, justify your answers and **show all your work**. If a box is provided, write your final answer there. **Unless otherwise indicated, simplification of numerical answers is required in these questions.**

- [8] 5. Using the definition of the derivative, compute $f'(x)$ if

$$f(x) = \frac{1}{x^2 - 1}$$

No marks will be given for the use of differentiation rules, but you may use them to check your answer.

[14] **6.** Let

$$f(x) = \frac{x^2}{x^2 + 1}$$

- (a) Determine the interval(s) where $f(x)$ is increasing, and the interval(s) where $f(x)$ is decreasing.
- (b) Find the x -coordinates of the local maxima and local minima of $f(x)$.
- (c) Determine the interval(s) where $f(x)$ is concave up and concave down, respectively.
- (d) Find the x -coordinates of all inflection points of $f(x)$.

parts (e) and (f) on next page...

(e) Find all asymptotes of the graph $y = f(x)$.

(f) Sketch the graph of $y = f(x)$, exhibiting all features in parts (a) – (e) and showing the y -coordinates of any points you found in parts (b) and (d).

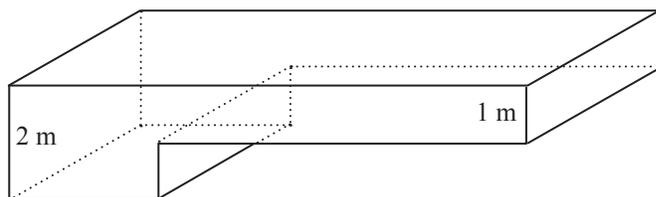
- [10] 7. A cup of coffee is left in a room that has a constant temperature of 25°C . The coffee is initially at 65°C . After it is left 15 minutes it is at 37°C . How long after it was put in the room was it at 50°C ? Assume that the temperature of the coffee cup obeys Newton's law of cooling. Any correct form of the final answer will be accepted; it does not need to be simplified completely.

Answer

- [10] 8. Consider a water tank shaped like an inverted right circular cone of height 1 m and top radius 1 m (the vertex of the cone is pointing downward). Let $h = h(t)$ denote the height of the water level of the tank in meters at time t in days. Water flows into the tank at a constant rate of 10 litres/day (1 litre = 10^{-3} m³). Water evaporates from the tank at a rate of $0.01A$ in m³/day where A is the area in m² of the water surface. When $h = 0.2$ m, how fast is the water level rising? *Note:* The volume of a right circular cone of height h and radius r is $\pi r^2 h/3$.

Answer

- [10] 9. A swimming pool is to be constructed in the shape depicted below. It has a shallow end of depth 1 m and a deep end of depth 2 m, and the shallow end is twice as long as the deep end. The total volume of the pool is 800 m^3 . What should the width of the pool be so that the total area of the seven surfaces that are in contact with water when the pool is completely filled with water (i.e. all surfaces except the open top surface of the pool) is minimized?



Answer

[6] **10.** Consider functions $f(x)$ defined on the interval $[0,1]$ that have the following properties:

- (i) f is continuous in the interval $[0,1]$.
- (ii) f is differentiable in the interval $(0,1)$.
- (iii) $f(0) = 0$ and $f(1) = 1$.

Find the function defined on the interval $[0,1]$ that has these properties that makes the maximum of $|f'(x)|$ on the interval $(0,1)$ as small as possible. *Hint:* begin by sketching some graphs that satisfy the properties above. *Prove* that your answer is correct.