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Work Learn International Undergraduate Research Award

The Composition Of A Lie Group Element As The Product Of Other Elements

Omar Mrani Zentar under the supervision of George Bluman

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Research Motivation

This project was a continuation of work done by Deshin Finlay under the supervision of George Bluman in summer 2014. The topic explored is related to Lie groups and Lie algebras. Lie groups of transformations can be used to represent a variety of continuous transformations such as scalings, translations, rotations, etc. In this research we examined how one can determine some elements of Lie groups of transformations as a product of other elements based on properties of the associated Lie algebra. The results obtained have many important applications to control theory as they yield, for example, specific combinations of rotations and translations in x that can lead to translation in y in a minimal number of steps. Additionally, solutions to nonlinear systems of differential equations were found by looking at the research problem from two different perspectives.

Research Problem

Let L be a Lie algebra of finite dimension. It is well known that the commutator of any two elements of the Lie algebra is a linear combination of its basic elements. Suppose that for two of the basic elements of L (B_1 and B_2) the coefficient, in the commutator, associated with a third basic element (B_3) is non-zero. Then the question of interest is how can one obtain the Lie group element associated with B_3 from a composition of the product of the Lie group elements associated with B_1 and B_2, using a minimal number of steps. For three dimensional Lie algebras, the research problem is equivalent to solving for a, b, c, and d in the following equation:

$$e^{a(\varepsilon)B_1} e^{b(\varepsilon)B_2} e^{c(\varepsilon)B_1} e^{d(\varepsilon)B_2} = e^{\varepsilon B_3}$$

Equation 1

Where a, b, c, and d are continuous functions of ϵ everywhere except at 0.

Results

We were able to solve the research problem for all three and four dimensional Lie algebras mentioned in '*Classification and Identification of Lie Algebras*' (1), using two different approaches. The first method relies on using the matrix representation of the Lie Algebras. The second approach, on the other hand, consists of differentiating equation 1, deriving a system of differential equations for the unknown variables, and then verifying that the solutions to the differential equations are also solutions to Equation 1. The latter is done using either a matrix representation of the Lie algebra or an operator representation if it is known.

Future Work

Currently, I am taking MATH 449 with my supervisor, George Bluman, in order to write a paper that will present the results of our research.

References

(1) Snobl, L., & Winternitz, P. (2014) *Classification and Identification of Lie Algebras*. Montreal, QC: CRM monograph series.