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# Non-negativity of the Coefficients of the cd-index of Bruhat Intervals

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A Bruhat interval is a subgraph of a special type of labeled, directed graph called a Bruhat graph consisting of all paths between two elements in the Bruhat graph. A Bruhat graph can be obtained through various methods; an explanation of the combinatorial approach can be found in the book by Bjoerner and Brenti [1]. Because each directed edge is labeled by a reflection of the group, which are ordered by a chosen reflection order, one can say whether two successive edges in a path increase or decrease. Recording this in the non-commuting variables $A$, for ascents, and $D$, for descents, in the ring $\mathbb{Z} < A, D >$ each path recieves an index of $A$'s and $D$'s. Adding these monomials all together for every path in the interval, one obtains an AD-polynomial for the interval, an AD-index of sorts. It was shown that this polynomial in fact lies in the subring $\mathbb{Z} < c, d >$ where $c = A + D$ and $d = AD + DA$ [2].

The goal of the research was to prove that this special cd-polynomial associated to a Bruhat interval, the cd-index, has non-negative coefficients. Much of the project consisted of computing examples in the simple case of one of the Bruhat intervals of $S_4$ in order to see whether any of the ideas involving flips, many of which were grounded in shellings of polytopes, resulted in well-controlled behaviour that would be amenable to induction.

One of these attempts was to consider the paths whose final step was less than or equal to a chosen reflection. The "boundary" was then defined to be those paths which when the top two edges were flipped, no longer had the top edge being less than the chosen reflection. The hope was that the cd-polynomial of this "boundary" would have non-negative coefficients as the polynomials of successive boundaries would yield the total cd-index. It was hoped that the boundary would consist of an interval of paths so that, by induction, it would be non-negative. Checking for the example of $S_4$, however, showed that there was no such nice structure and the boundary approach was abandoned once it was realized that it had no immediately clear structure.

After multiple other purely combinatorial attempts also proved fruitless, the conclusion was come to that perhaps one may have to appeal to the geometric properties of

Bruhat intervals. The difficulty then remains to find a good geometric interpretation of the cd-index. Conjectures towards this have been made as mentioned in [3].

## References

[1] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter Groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, April 2005.

[2] K. Karu. On the complete cd-index of a Bruhat interval. *ArXiv e-prints*, January 2012.

[3] K. Karu. M-vector analogue for the cd-index. *ArXiv e-prints*, December 2014.