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SPECTRA OF SOL MANIFOLDS

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This summer, I worked with Professors Malabika Pramanik, Mahta Khosravi and fellow student Curt da Silva to study the spectrum of the Laplace-Beltrami operator on a special 3-dimensional manifold called the Sol manifold. The first part of our project was devoted to understanding the language of basic differential and Riemannian geometry. Topics that we covered here included smooth manifolds, Riemannian manifolds, tangent spaces, Lie groups, Lie algebras, tensors, and integration of differential forms.

With this machinery in hand, we were then able to study the manifold of interest to our research, the Sol manifold. The Sol manifold was introduced by William Thurston in 1980 in his famous *Geometrization Conjecture* which stated that any compact orientable 3-dimensional manifold could be decomposed into submanifolds which admit one of 8 special geometric structures with one of these geometries being the *Sol* geometry. (For pure interest, it is worth noting that a proof to the *Geometrization Conjecture* actually implies the solution of a well celebrated problem in mathematics: the Poincare conjecture; a full proof of this was presented by Grigori Perelman in 2003) The Sol manifold is defined to be $\mathbb{T}^2$ torus bundles over a circle S^1 with hyperbolic gluing maps. More precisely, consider the action of $\mathbb{Z}$ on $\widetilde{M}^3 = \mathbb{T}^2 \times \mathbb{R}$ generated by the transformation T_A defined as:

$$T_A : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \\ z+1 \end{pmatrix}$$

where $(x, y) \in \mathbb{T}^2$ taken modulo 1, $z \in (-\infty, \infty)$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL(2, \mathbb{Z})$. The Sol manifold M_A^3 is defined as the quotient $\widetilde{M}^3/\mathbb{Z}$ under the action of T_A.

Our prime interest was in studying the spectrum of the Laplace-Beltrami operator on the Sol manifold. The importance of this arises from the fact that the spectrum gives us important information about the geometry of the Sol manifold. It has been shown in the paper "Spectra of Sol Manifolds" by Bolsinov, Dullin, Veselov that for the Laplace-Beltrami operator, we have the following formula:

$$N(\lambda) \sim \frac{4}{3}\pi\lambda^{\frac{3}{2}}\frac{Vol(M_A^3)}{(2\pi)^3} \tag{0.1}$$

where $N(\lambda)$ is the number of eigenvalues $\mathcal{E} \leq \lambda$ and $Vol(M_A^3)$ is the volume of the Sol-manifold.

Explicitly, the spectrum of the Laplace-Beltrami operator consists of a trivial part and a non-trivial part. The trivial part are the eigenvalues $\mathcal{E} = \mathcal{E}_l = 4l^2\pi^2$ corresponding to eigenfunctions $e^{2\pi i l z}$, where $l = 0, 1, \dots$; the non-trivial part is much more complicated and is related to the modified Mathieu equation. Another significant difference between the trivial part of the spectrum and the non-trivial part is the multiplicities of the eigenvalues. It is easy to see that for the trivial part, the multiplicities are 2 except for $\mathcal{E}=0$ which has multiplicity 1. But for the non-trivial part, the multiplicities are much more complicated but interesting. In fact, computing these multiplicities can be interpreted as the number theoretic problem of finding the number of integer solutions of primitive indefinite quadratic forms.

The main research problem that we worked on was conducting a numerical study of the eigenvalue counting function $N(\lambda)$; our main goal was to check computationally that the the formula (0.1) holds. At this current moment, our numerical analysis shows that the eigenvalue counting function $N(\lambda)$ has the asymptotic behaviour $N(\lambda) \sim \lambda^{\frac{5}{4}}$ which is a somewhat off as we would want that $N(\lambda) \sim \lambda^{\frac{3}{2}}$ as suggested by (0.1). It is hoped, perhaps after some modifications, that our numerical analysis of the asymptotic behaviour of $N(\lambda)$ will correspond with that given by (0.1).