SUMMER 2017 - NSERC/WL USRA REPORT
Computing Mean First Passage Time On Surfaces With Traps

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1 Introduction

The first passage time problem based on Brownian particles has wide applications in biology and finance, such as cellular signal processing [1] and modeling the optimal time to sell an asset [2]. Some recent studies have focused on computing mean first passage time (MFPT) with stationary traps on a 3-D sphere [3] and a moving trap on a 2-D disk [4], but traps on an ellipsoid or arbitrary triangulate surfaces in 3-D has not been explored deeply yet. Therefore, we present an algorithm for computing the MFPT on 3-D surfaces with a trap by solving a PDE. Our algorithm uses the Closest Point Method (CPM) [5] for dealing with a potentially curved substrate. The highlight of this algorithm is to modify the surface by modifying its original closest point function of the surface. We also demonstrate our algorithm with several illustrative computations and comparisons. The results will have applications in many areas, especially in biological system.

2 The method of cutting holes on 3-D surfaces

Suppose we have a surface represented by its closest point function. Our aim is to put a trap on the surface by cutting a hole on it. We would like to modify the surface by modifying the closest function of the surface. Then, we will use the modified closest function for the punctured domain to compute the MFPT for a trap.

We use a sphere to cut a hole on the 3-D surface and assume the hole is between the intersection of the surface and the sphere. We formulated the function by finding the closest points when there is a hole on the surface, based on the Brouwer fixed point Iteration theorem [6]. It states that for any continuous function $f(x)$ mapping a compact convex set into itself, there is a point x such that $f(x) = x$. The approximated solution sequence will converge to the exact solution. Let $CP_{surface}$ and CP_{sphere} denote the closest point function on the surface and the sphere respectively. Let $f(z) = CP_{surface}(CP_{sphere}(z))$ or $CP_{surface} \circ CP_{sphere}$. Given a starting point x, $z_0 = CP_{surface}(x)$. Our fixed point iteration is $f(z) = z$, $z_{k+1} = f(z_k)$. Finally, we want $\lim_{k \to +\infty} z_k \to CP(x)$. We also apply the surface area iteration to pick the estimated intersection area that is closest to the given area of the trap.

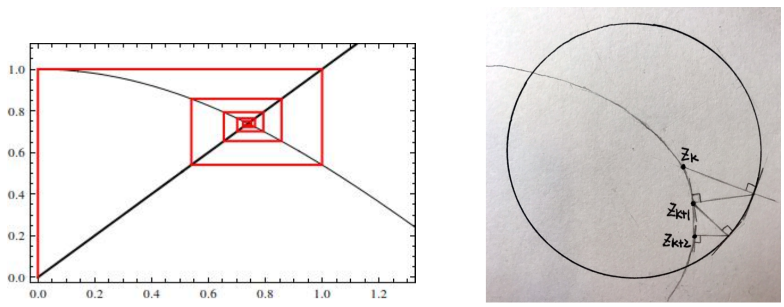

Figure 1: Illustrations of the fixed point iteration (the right one is drawn by the author).

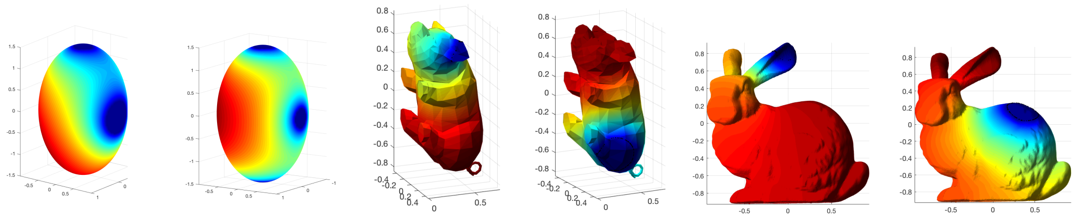

Figure 2: The diffusion pattern with holes on ellipsoids and some triangulate surfaces: pig and Stanford bunny in 3-D.

3 Some results of MFPT on a 3-D ellipsoid with a trap

Consider a Brownian particle on the surface of an ellipsoid with major axis 1.5 and minor axis 1 in 3-D. The surface has a stationary absorbing trap whose surface area is approximately 0.1. The MFPT for the particle satisfies the Poisson equation $\Delta_{\mathcal{S}} u = -1/D$ [7], where the diffusion coefficient is $D = 1$ and the solution depends on the location of the trap. The research question is to figure out the optimal trap location that minimizes the capture time for such Brownian particles. Let θ be the polar angle of the trap center. The global minimum of maximum MFPT occurs at the equator where $\theta = 0$. When Δx was decreased to 0.05, the plot is smoothly convergent.

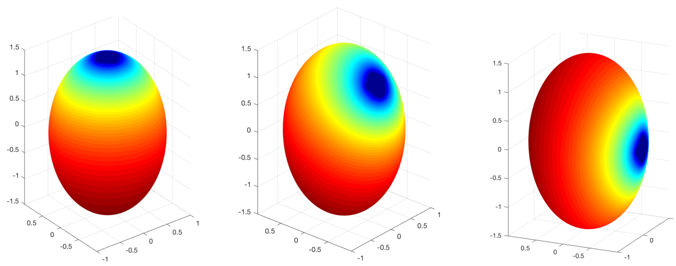

Figure 3: The diffusion patterns after cutting holes on the ellipsoid in 3-D where $\theta = \pi/2$, $\pi/4$ and 0 respectively. The blue region is where the trap located.

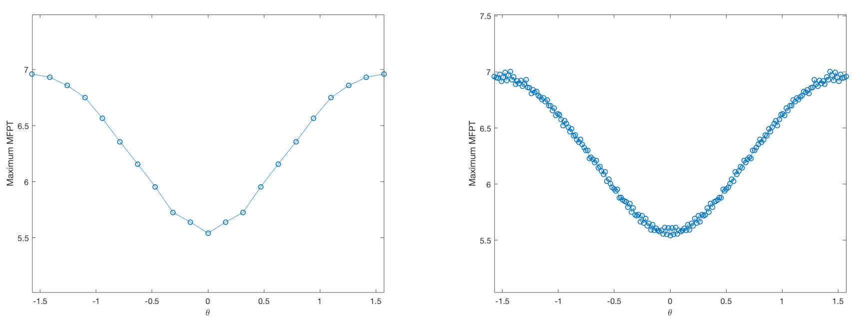

Figure 4: Maximum MFPT against θ with 21 (left) and 201 (right) data points with the grid size $\Delta x = 0.05$.

Inspired by the biodiversity that cells can have various shapes, we also examined our algorithm on different shapes of ellipsoid. As the minor axis decreases, the difference between the global maximum and the global minimum of the maximum MFPT increases.

In conclusion, on our ellipsoid surface, MFPT increases if the particle starts farther away from the trap. Moreover, we should put the trap on the lowest curvature of the ellipsoid to minimize the overall capture time.

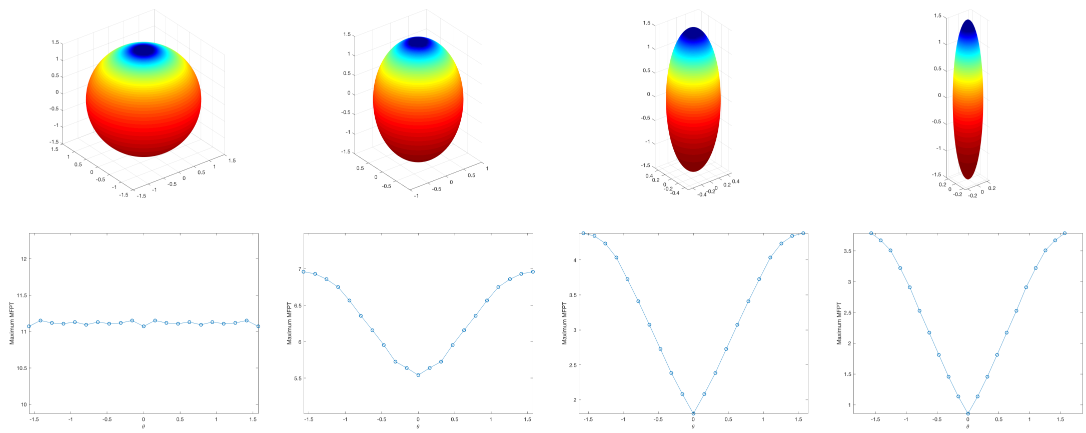

Figure 5: Maximum MFPT against θ from -$\pi/2$ to $\pi/2$ with 21 data points and $\Delta x = 0.05$ on four different ellipsoids with same major axis value: 1.5, but different values of minor axis: 1.5, 1, 0.5, 0.25 respectively and approximately the same surface area of the trap: 0.1.

4 Acknowledgement

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References

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