Investigations on the spectrum of the Laplacian on the Cayley graph associated with triangle groups generated by reflections in the hyperbolic plane

The spectrum of the graph Laplacian is of interest to both mathematicians and physicists alike. In the Poincare disk model of the hyperbolic plane, we define our base triangle(i.e. fundamental domain) to be the the triangle with the angle pi/2 located at the origin and the other two angles being pi/3 and pi/7. We denote reflections across the 3 sides of the triangle to be a, b and c, which then generates the group of isometries G, and we also have the Cayley graph associated with G with respect to the reflection generators.

Our goal is to study the spectrum of the Laplacian $\sigma(\Delta_G)$ (or equivalently of the adjacency operator A) on the Cayley graph, which is a regular graph of degree 3. We may write $A = T_a + T_b + T_c$, where T_a, T_b, T_c are the operators on $l^2(G)$ associated to a, b and c by the right regular representation, and we showed that $\sigma(T_a + T_b + T_c) = \sigma(-\Delta_G \otimes I)$. We focus our study on a closely related representation on L^2 of the hyperbolic plane.

The investigation begins with the (2,4,4) triangle group in the Euclidean space to illustrate the ideas and warm up for the (2,3,7) triangle group. Some computational results are shown throughout our investigation (Matlab plots of eigenvalues vs different parameters). Then, we introduce the hyperbolic case with the (2,3,7) triangle group and numerically approximate the spectrum using Matlab.

After reaching our initial goal, we move on to investigate other triangle groups in the hyperbolic case using similar methods. We test wether the angle sum of the base triangle completely determines the spectrum (or at least its maximum) and the answer seems to be negative. We also study the (∞, ∞, ∞) triangle group and the (q, ∞, ∞) triangle group, and our numeric approximation of the maximum of the spectrum for the (∞, ∞, ∞) triangle group confirms the established value of $2\sqrt[2]{2}$.