

NSERC USRA Report – Summer 2013

Resolution of Singularities and Toric Varieties

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1 Introduction

During the summer, I had the privilege and pleasure of studying under supervision of Professors Julia Gordon and Sujatha Ramdorai on the resolution of singularities and related topics in algebraic geometry. Very roughly speaking, the goal is to take some singular algebraic variety X (for example, a curve or surface), and to resolve ("smooth out") its singular points, which in some sense are "badly" behaved points, and at the end of the process to obtain a new variety \tilde{X} (called desingularization, or nonsingular model of X) which is birationally isomorphic to X away from the singular points. In order to give more precise formulation of the problem, one needs to introduce the concept of blow-up. In simplest terms, blow-up is the process of replacing a singular point with the projectivized tangent space. For example, one can blow-up the origin of the affine plane \mathbb{A}^2 as depicted in the diagram below.

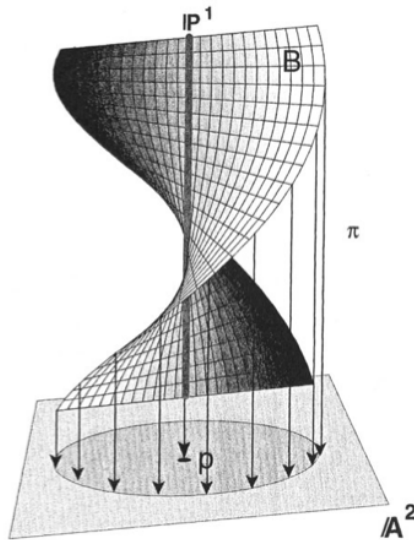


Figure 1: Blow-up of the plane at a point

Following [5], we give the definition of a blow-up at the origin of \mathbb{A}^n . Consider

$$B = \{(x, \ell) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : x \in \ell\} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$$

The set B is called **one point blow-up** of \mathbb{A}^n at the origin. We have the canonical projection:

$$\pi : B \rightarrow \mathbb{A}^n$$

by sending (x, ℓ) to x . Let's investigate the fibers of π . If $x \in \mathbb{A}^n \setminus \{0\}$, then $\pi^{-1}(x)$ is the unique line through x and the origin, which we can denote by ℓ_x . So if $x \neq 0$, then $\pi^{-1}(x) = (x, \ell_x)$. However, since every $\ell \in \mathbb{P}^{n-1}$ passes through the origin, we have $\pi^{-1}(0) = (0, \mathbb{P}^{n-1})$. An intuitive way to visualize the blow-up is that the map $\pi : B \rightarrow \mathbb{A}^n$ collapses the entire projective space \mathbb{P}^n to the origin, but is bijective elsewhere. The Figure 1 is exactly the situation described when $n = 2$.

One important application of blow-up is that if we look at the inverse image of a singular curve under the aforementioned map π , the resulting new curve (now living in $\mathbb{A}^2 \times \mathbb{P}^1$) may no longer possess the singularity, which is what we want! Consider the example below, where singular point $(0, 0)$ of $y^2 = x^3 + x^2$ has been unknotted.

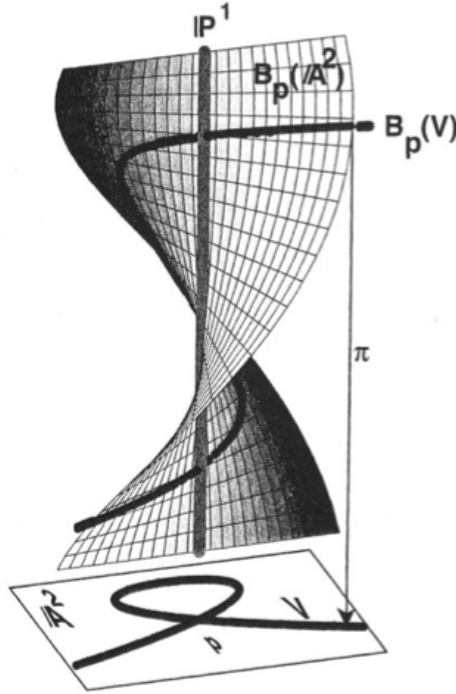


Figure 2: Blow-up of the nodal curve $y^2 = x^3 + x^2$ at the nodal point

The precise formulation of the problem for resolving singularities of algebraic varieties, sometimes called strong desingularization, asks for a sequence of blow-ups

that satisfy certain conditions, in particular, requiring that the exceptional divisor (i.e. the inverse image of the centre of the blow-up) is of particularly nice form, e.g. has normal crossings. What this means exactly requires the concept of normal crossing divisor, and it would take this report too far afield to define such a term. In any case, the complete answer was given to this question in the affirmative when k (the ground field) has characteristic zero by Hironaka [4].

2 Toric Varieties

The so-called toric varieties have an interesting combinatorial interpretation, and in this setting resolution of singularities has been studied extensively. A toric variety is defined as an irreducible variety X containing a torus $T_N \cong (\mathbb{C}^*)^n$ as a Zariski open subset, such that the action of T_N on itself extends to an algebraic action of T_N on X . This is a rather abstract definition, but it turns out that toric varieties also arise from concrete combinatorial data, such as cones and fans. For a finite set of vectors $S \subseteq \mathbb{R}^n$, we define the **convex polyhedral cone** spanned by S to be the set of all linear combination of vectors in S with positive coefficients in \mathbb{R} . A finite collection \mathcal{F} of polyhedral cones is called a **fan** if $\tau \in \mathcal{F}$ for every face τ of every $\sigma \in \mathcal{F}$, and $\sigma_1 \cap \sigma_2 \in \mathcal{F}$ for every $\sigma_1, \sigma_2 \in \mathcal{F}$. For example, the projective plane \mathbb{P}^2 and a product of two copies of projective line $\mathbb{P}^1 \times \mathbb{P}^1$ can be realized as:

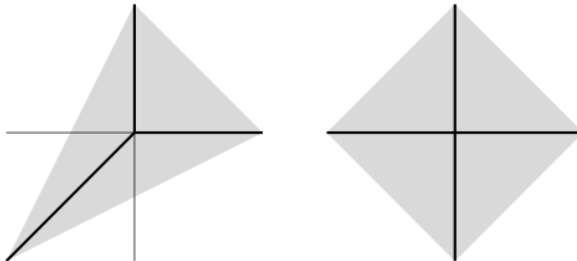


Figure 3: Fans for \mathbb{P}^2 (left) and $\mathbb{P}^1 \times \mathbb{P}^1$ (right)

There is a beautiful connection between resolution of singularities and continued fractions in the case of two-dimensional affine toric varieties [7]. It can be shown that any singular two-dimensional affine toric variety can be constructed from a cone σ generated by two vectors $v_1 = e_2 = (0, 1)$ and $v_2 = me_1 - ke_2 = (m, -k)$ where $0 \leq k < m$ and $\gcd(m, k) = 1$. To resolve the singularities, one can use the idea of "fan refinement" which means inserting a new ray into the middle of our cone σ . The main observations are: 1) The vectors e_1 and e_2 generate the full lattice \mathbb{Z}^2 , so the resulting toric variety is non-singular, and 2) Inserting extra vectors to singular toric variety corresponds to blowing-up. Based on these two ideas one can prove that a sequence of refinements of the cone will correspond to a sequence of blow-ups leading to a complete desingularization of the corresponding toric variety. More precisely, suppose a cone σ is given in the standard form (i.e. generated by e_2 and $me_1 - ke_2$,

$0 \leq k < m$, $\gcd(m, k) = 1$). We insert the ray $e_1 = (1, 0)$ into σ . The "upper cone" generated by e_1 and e_2 generates non-singular variety, and the "lower" cone between the rays is now generated by e_1 and $me_1 - ke_2$.

We can rotate the lattice by 90° counter-clockwise, which will bring the lower cone into standard position. We also need to translate the vector (m, k) to its standard location, which will be $(m_1, -k_1)$ where $m_1 = k$, $0 \leq k_1 < m_1$ and $k_1 = a_1k - m$ for some $a_1 \geq 2$ (existence of a_1, k_1 is guaranteed by division algorithm).

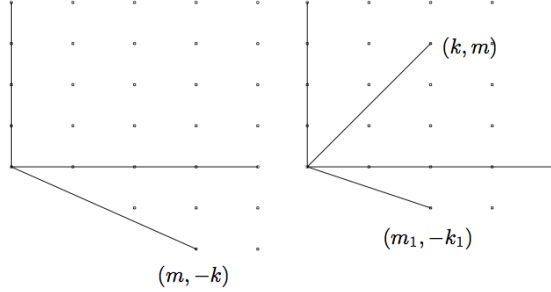


Figure 4: Blow-up for a 2-dimensional Affine Toric Variety

Now if $k_1 = 0$, then the cone would be smooth because it is generated by a basis. Otherwise, we have

$$\frac{m}{k} = \frac{a_1k - k_1}{m_1} = a_1 - \frac{k_1}{m_1} = a_1 - \frac{1}{\frac{m_1}{k_1}}$$

We can repeat the process (until $k_r = 0$) to get:

$$\frac{m}{k} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\dots - \frac{1}{a_r}}}}$$

If we recursively define $v_{i+1} = a_i v_i - v_{i-1}$ with initial rays $v_0 = e_2$ and $v_1 = me_1 - ke_2$, then we have inserted r rays into our original cone. The procedure has r successive blow-ups, and so there will be r exceptional divisors, say E_i for $1 \leq i \leq r$. Furthermore, $E_i \cong \mathbb{P}^1$ for each i . More surprisingly, E_i and E_j will intersect transversally for each $i \neq j$, and self-intersection number will be $E_i \cdot E_i = -a_i$. The interpretation for negative self-intersection involves the concept of normal bundles. See Figure 5.

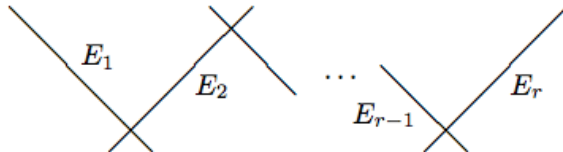


Figure 5: Dual graph for exceptional divisors

3 My role

Since my background in commutative algebra wasn't strong at the start of the summer, I started working through the classic "Introduction to Commutative Algebra" by Atiyah & Macdonald [1]. I solved good amount of the exercises (typed up in 53 page long document) in the first 8 chapters, which Professor Ramdorai and her grad student Li Zheng kindly checked and provided me with great feedback. I also completed reading some basic algebraic geometry texts, such as [5].

I tried to use Macaulay2 and Singular software to understand resolution of singularities for variety of nilpotent elements in a Lie algebra, and other closely related varieties. An example of such a variety would be the set of all nilpotent matrices in $M_{n \times n}(\mathbb{C})$, viewed as a closed subset in the Zariski topology. As a result, we can investigate its algebro-geometric properties such as its dimension, singular locus, etc. One tentative goal was that perhaps we could borrow techniques for explicit resolution of singularities from the study of toric varieties to the case of nilpotent varieties. We are aware of the Springer resolution [6], but it is not yet clear how to think of it as a blow-up, and we would have liked to have exceptional divisors with normal crossings. While I haven't concluded anything worthwhile in this direction, learning how to use these computer algebra systems has been greatly beneficial. I have decided to continue examining some of the unanswered questions during the school year.

I am most grateful to have been awarded NSERC USRA for the summer, which has contributed vastly to my mathematical development, and has given me extra motivation to go further in learning algebraic geometry (which I plan to do in graduate school). I will take this opportunity to thank Professors Julia Gordon and Sujatha Ramdorai for spending their many hours for elucidating some of the finest points in commutative algebra/algebraic geometry, and the explanations they have provided have been more helpful than any textbook could offer. I also want to thank Math department IT support, especially Thi, for kindly helping me to install Singular software and its graphics support to my computer.

4 Acknowledgements

Figures 1 and 2 have been taken from [5]. Figure 3 has been taken from [2]. And Figures 4 and 5 have been taken from [7].

References

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