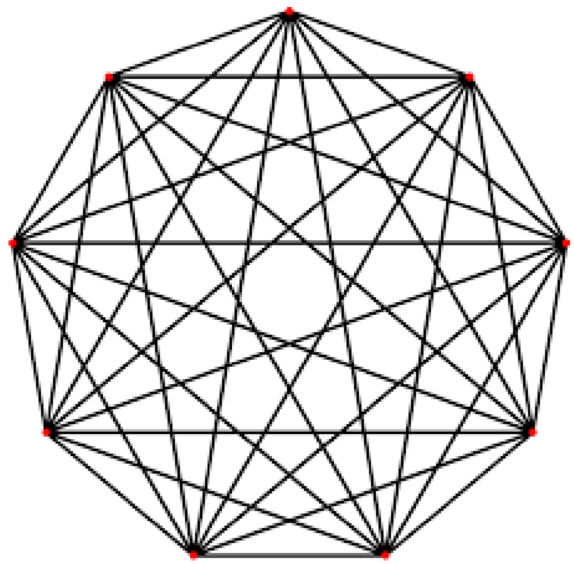
LECTURE 4

Curie-Weiss model. Phase transition.
Metastable regime, Eyring-Kramers formula.



Complete graph with $N = 9$ vertices

GLAUBER DYNAMICS ON THE COMPLETE GRAPH

Let $G = (V, E)$ be a connected graph. Ising spins are attached to the vertices V and interact with each other along the edges E.

1. The energy associated with the configuration $\sigma = (\sigma_i)_{i \in V} \in \Omega = \{-1, +1\}^V$ is given by the Hamiltonian

$$H(\sigma) = -J \sum_{(i,j) \in E} \sigma_i \sigma_j - h \sum_{i \in V} \sigma_i$$

where $J > 0$ is the ferromagnetic interaction strength and $h > 0$ is the external magnetic field.

2. Spins flip according to Glauber dynamics

$$\forall\, \sigma \in \Omega\ \forall\, j \in V: \quad \sigma \to \sigma^j \text{ at rate } \mathrm{e}^{-\beta[H(\sigma^j)-H(\sigma)]_+}$$

where σ^j is the configuration obtained from σ by flipping the spin at vertex j, and $\beta > 0$ is the inverse temperature.

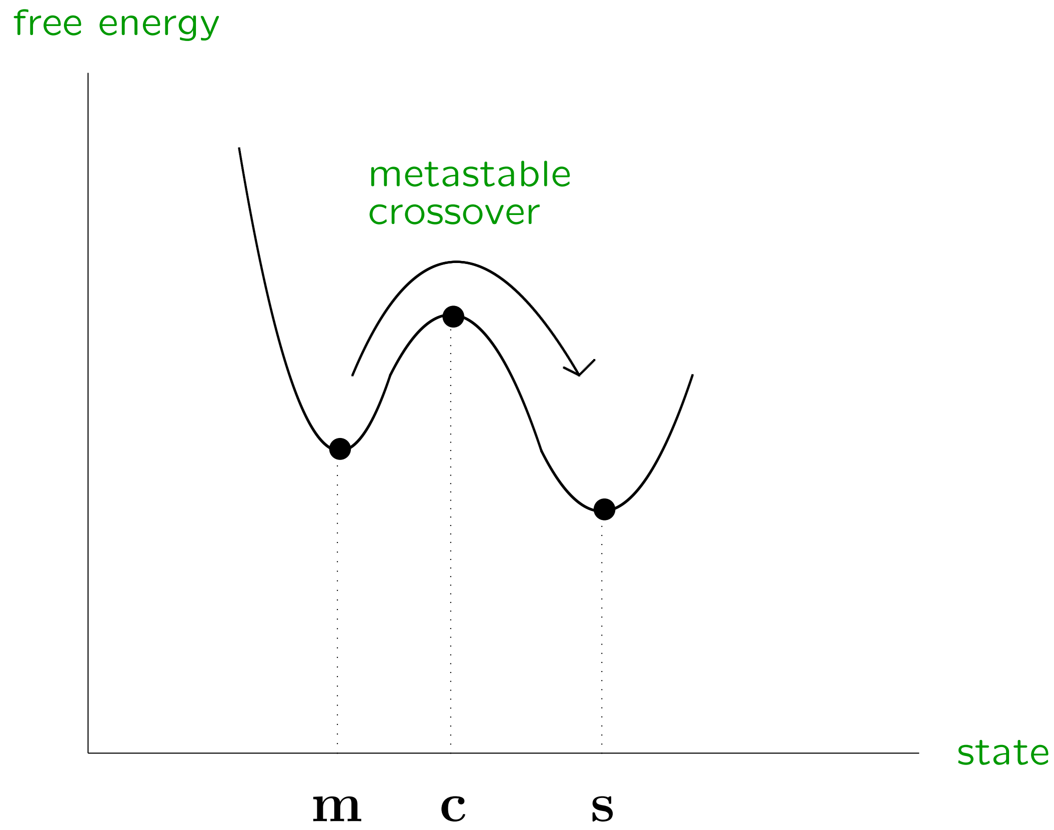
3. The Gibbs measure

$$\mu(\sigma) = \frac{1}{\Xi}\,\mathrm{e}^{-\beta H(\sigma)}, \qquad \sigma \in \Omega,$$

is the reversible equilibrium of $(\sigma(t))_{t\geq 0}$.

4. Three sets of configurations play a central role:

$$\mathbf{m} = \text{metastable state}$$
$$\mathbf{c} = \text{crossover state}$$
$$\mathbf{s} = \text{stable state.}$$



Caricature of the free energy landscape

– energy and entropy –

In the Hamiltonian it is natural to pick $J = 1/N$ with $N = |V|$, to ensure that the total interaction of a spin with all the other spins is of order 1.

Because the interaction is mean-field, we can implement a lumping technique in which we monitor the magnetisation $m \in [-1, +1]$ of the system rather than the configuration $\sigma \in \Omega$.

§ EVOLUTION OF THE MAGNETISATION

The lumping shows that the empirical magnetisation

$$m_N(t) = \frac{1}{N} \sum_{i \in V} \sigma_i(t)$$

performs a continuous-time random walk on the $2N^{-1}$-grid in $[-1,1]$, in a potential that is given by the finite-volume free energy per vertex

$$f_{\beta,h,N}(m) = -\tfrac{1}{2}m^2 - hm + \beta^{-1} I_N(m)$$

with an entropy term

$$I_N(m) = -\frac{1}{N} \log \binom{N}{\frac{1+m}{2}N}.$$

EXERCISE!

This simplification arises via the mean-field interaction:

$$H(\sigma) = -N\Big[\tfrac{1}{2}(m_N(t))^2 - \tfrac{1}{2} + h m_N(t)\Big].$$

In the limit $N \to \infty$, the empirical magnetisation performs a Brownian motion on $[-1,+1]$, in a potential that is given by the infinite-volume free energy per vertex

$$f_{\beta,h}(m) = -\tfrac{1}{2}m^2 - hm + \beta^{-1} I(m)$$

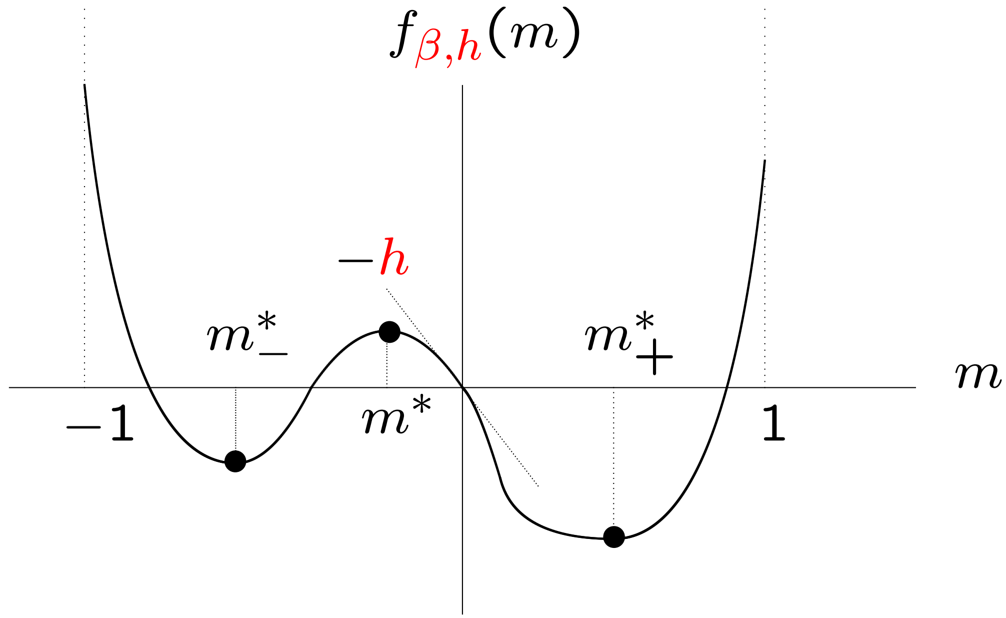
with

$$I(m) = \tfrac{1}{2}(1+m)\log(1+m) + \tfrac{1}{2}(1-m)\log(1-m),$$

where a redundant shift by $-\log 2$ is dropped.

Kramers

EXERCISE!



The free energy per vertex $f_{\beta,h}(m)$ at magnetisation m
(caricature picture with $\mathbf{m} = m^*_-$, $\mathbf{c} = m^*$, $\mathbf{s} = m^*_+$).

EXERCISE!

THEOREM 4.1: Bovier, Eckhoff, Gayrard, Klein 2001

On the complete graph with N vertices, for $J = 1/N$, $\beta > 1$ and $h \in (0, \chi(\beta))$,

$$\mathbb{E}^{\mathrm{CW}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+}) = K\,\mathrm{e}^{N\Gamma}[1+o(1)], \qquad N \to \infty,$$

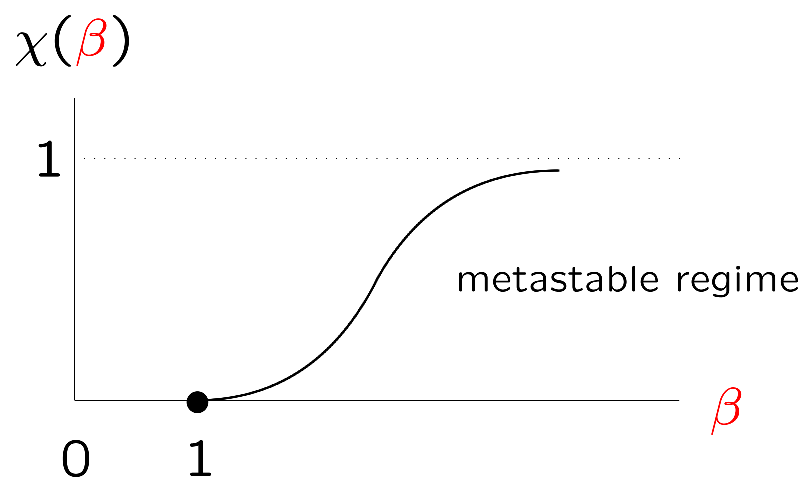
where $\mathbf{m}_N^-, \mathbf{m}_N^+$ are the sets of configurations for which the magnetization tends to m_-^*, m_+^*,

$$\Gamma = \beta\,[f_{\beta,h}(m^*) - f_{\beta,h}(m_-^*)]$$

$$K = \pi\beta^{-1}\sqrt{\frac{1+m^*}{1-m^*}\,\frac{1}{1-m_-^{*2}}\,\frac{1}{[-f''_{\beta,h}(m^*)]f''_{\beta,h}(m_-^*)}}$$

and

$$\chi(\beta) = \sqrt{1-\tfrac{1}{\beta}} - \tfrac{1}{2\beta}\log\left[\beta\left(1+\sqrt{1-\tfrac{1}{\beta}}\right)^2\right].$$



The conditions on β and h are needed to ensure that the free energy $m \mapsto f_{\beta,h}(m)$ has a double-well structure.

NOTE: The asymptotics for the crossover time is uniform in the starting configuration drawn from the set $\mathbf{m}_N^-$.

§ CURIE-WEISS IN DISCRETE TIME

1\. Lumping works well for the Curie-Weiss model. The state space is $S_N \equiv \{-1,+1\}^{[N]}$ with $[N] = \{1,\dots,N\}$, $N \in \mathbb{N}$. The Hamiltonian is (add the diagonal term $\frac{1}{2}$)

$$H_N(\sigma) \equiv -\frac{1}{2N} \sum_{i,j\in[N]} \sigma_i\sigma_j - h \sum_{i\in[N]} \sigma_i, \qquad \sigma \in S_N.$$

The fact that the model is mean-field is expressed by the fact that $H_N(\sigma)$ depends on σ only through the empirical magnetisation

$$m_N(\sigma) \equiv \frac{1}{N} \sum_{i\in[N]} \sigma_i,$$

namely,

$$H_N(\sigma) = -N\left(\tfrac{1}{2}m_N^2(\sigma) + h m_N(\sigma)\right) \equiv N E(m_N(\sigma)).$$

2. Choose a discrete-time dynamics on S_N with Metropolis transition probabilities

$$p(\sigma,\sigma') = \begin{cases} N^{-1}\,\mathrm{exp}\left[-\beta\left[H_N(\sigma')-H_N(\sigma)\right]_+\right], & \|\sigma-\sigma'\|_1 = 2, \\ 0, & \|\sigma-\sigma'\|_1 > 2, \\ 1-\sum_{\eta\neq\sigma} p(\sigma,\eta), & \sigma=\sigma', \end{cases}$$

where $\|\cdot\|_1$ is the ℓ^1-norm on S_N, and the last line is put in to obtain a proper normalisation. This dynamics is reversible w.r.t. the Gibbs measure

$$\mu_N(\sigma) = \frac{1}{Z_{\beta,N}}\,\mathrm{e}^{-\beta H_N(\sigma)}\,2^{-N}, \qquad \sigma\in S_N.$$

3. The magnetisation $m_N(n) \equiv m_N(\sigma(n))$ at time $n \in \mathbb{N}_0$ can only increase or decrease by $2N^{-1}$, and the probability of doing so only depends on the number of -1's and $+1$'s in the configuration $\sigma(n)$, namely,

$$\mathbb{P}\left(m_N(n+1) = m' \mid \mathcal{F}_n\right) = r_N(m_N(n), m'), \qquad n \in \mathbb{N}_0,$$

with $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ the natural filtration and

$$r_N(m,m') = \begin{cases} \frac{1-m}{2} \exp\left[-\beta N[E(m') - E(m)]_+\right], & m' = m + 2N^{-1}, \\ \frac{1+m}{2} \exp\left[-\beta N[E(m') - E(m)]_+\right], & m' = m - 2N^{-1}, \end{cases}$$

on the state space

$$\Gamma_N \equiv \left\{-1, -1+2N^{-1}, \ldots, 1-2N^{-1}, 1\right\}.$$

4. The Markov process is reversible with respect to the image Gibbs measure

$$\nu_N(m) = \frac{1}{\bar{Z}_{\beta,N}} \mathrm{e}^{-\beta N E(m)} \binom{N}{\frac{1+m}{2}N} 2^{-N}, \qquad m \in \Gamma_N.$$

In exponential form the latter can be written as

$$\nu_N(m) = \mathrm{e}^{-\beta N \bar{f}_N(m)},$$

where

$$\bar{f}_N(m) = f_{\beta,N}(m) - \inf_{m \in \Gamma_N} f_N(m),$$
$$f_N(m) = -\tfrac{1}{2}m^2 - hm + \beta^{-1} I_N(m),$$

with

$$-I_N(m) = \frac{1}{N} \log \left[\binom{N}{\frac{1+m}{2}N} 2^{-N} \right].$$

5. In the limit as $N \to \infty$, Γ_N lies dense in $[-1,1]$, and

$$\lim_{N\to\infty} f_N(m) = f(m),$$
$$\lim_{N\to\infty} I_N(m) = I(m),$$

with

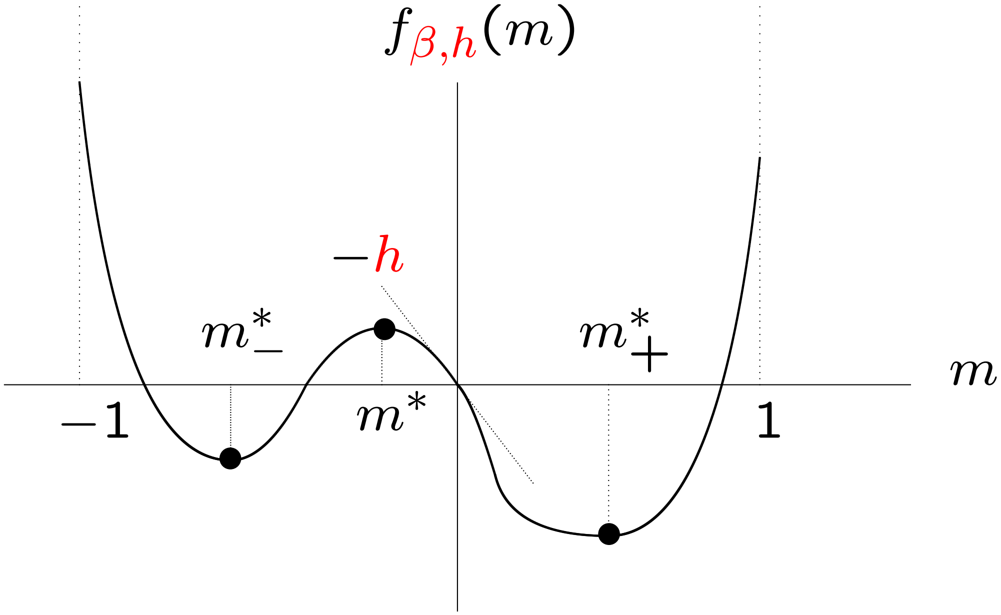
$$f(m) = -\tfrac{1}{2}m^2 - hm + \beta^{-1} I(m),$$
$$I(m) = \tfrac{1}{2}(1+m)\log(1+m) + \tfrac{1}{2}(1-m)\log(1-m).$$

The latter is the Cramér rate function for coin tossing.

Since $I(m) = I(-m)$ and $I(m) \sim \frac{1}{2}m^2$ as $m \to 0$, we see that $m \mapsto f(m)$ is a double well when $\beta > 1$ and h is small enough. The stationary points of f are the solutions of the equation

$$m = \tanh[\beta(m+h)],$$

which we refer to as the self-consistency equation.

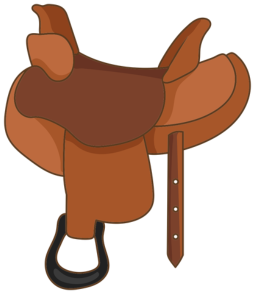


Plot of $m \mapsto f_{\beta,h}(m)$ on $[-1,1]$ when $\beta > 1$ and $h \in (0, \chi(\beta))$.

Henceforth we suppress β, h from the notation.

6. The above observations show that $m_N = (m_N(n))_{n\in\mathbb{N}_0}$ is a random walk on $\Gamma_N \subset [-1,1]$ with a reversible invariant measure that is close to $\exp[-\beta f(m)]$ for large N.

Clearly, this bring us to a situation where we can obtain an exact solution. Moreover, since we are on a lattice with spacing $2/N$, in the limit as $N \to \infty$ sums appearing in the exact solution can be approximated by integrals with the help of saddle-point techniques.



§ PROOF OF THEOREM 4.1

1. Since $(m_N(t))_{t\geq 0}$ is a nearest-neighbour random walk on Γ_N, we have, as explained in Lecture 2,

$$\mathbb{E}_{m_N^-}\left[\tau_{m_N^+}\right] = \sum_{\substack{m,m'\in\Gamma_N \\ m_N^- < m \leq m_N^+,\, m'\leq m}} \frac{\nu_N(m')}{\nu_N(m)} \frac{1}{r_N(m,m+2N^{-1})}.$$

Compute the summand:

$$r_N(m,m+2N^{-1}) = \tfrac{1-m}{2}\,\mathrm{e}^{-\beta N[E(m+2N^{-1})-E(m)]_+},$$

$$\frac{\nu_N(m')}{\nu_N(m)} = \mathrm{e}^{\beta N[f_N(m)-f_N(m')]}.$$

In the limit as $N \to \infty$, the sums over m, m' are dominated by the terms with $m \to m^*$ and $m' \to m^*_+$, since for these terms $f_N(m) - f_N(m')$ is maximal. This explains the exponential factor in the Eyring-Kramers formula.

2. To get the prefactor we need to look a bit more closely. Note that

$$[E(m + 2N^{-1}) - E(m)]_+ = 2N^{-1}[(m + h) + N^{-1}]_+.$$

For $m \to m^*$ and $N \to \infty$, $r_N(m, m + 2N^{-1})$ converges to

$$\frac{1 - m^*}{2} \exp(-2\beta[m^* + h]_+).$$

As shown in the figure: $m^* < 0$. Since m^* is a solution of the self-consistency equation

$$\exp(2\beta[m^* + h]) = (1 + m^*)/(1 - m^*),$$

it follows that $m^* + h < 0$. Therefore, for any $\epsilon > 0$,

$$\mathbb{E}_{m_N^-}\left[\tau_{m_N^+}\right]$$
$$= \mathrm{e}^{\beta N[f_N(m^*) - f_N(m^*_-)]}\, \frac{2}{1-m^*}\, [1+o(1)] \times$$
$$\sum_{\substack{m,m' \in \Gamma_N \\ |m-m^*|<\epsilon,\, |m'-m^*_-|<\epsilon}} \mathrm{e}^{\beta N[f_N(m) - f_N(m^*)] - \beta N[f_N(m') - f_N(m^*_-)]}.$$

3. By Stirling's formula, for any $m \in (-1, 1)$,

$$I_N(m) - I(m) = [1 + o(1)]\, \frac{1}{2N} \log\left(\frac{\pi N(1-m^2)}{2}\right)$$

and hence

$$\mathrm{e}^{\beta N[f_N(m) - f(m)]} = [1 + o(1)] \sqrt{\frac{\pi N(1-m^2)}{2}}.$$

Consequently,

$$\mathrm{e}^{\beta N[f_N(m^*)-f_N(m^*_-)]}$$

$$= [1+o(1)]\,\mathrm{e}^{\beta N[f(m^*)-f(m^*_-)]}\sqrt{\frac{1-m^{*2}}{1-m_-^{*2}}}.$$

Inserting this expansion into the earlier expression, we get

$$\mathbb{E}_{m_N^-}\left[\tau_{m_N^+}\right]$$

$$= \mathrm{e}^{\beta N[f(m^*)-f(m^*_-)]}\,\frac{2}{1-m^*}\sqrt{\frac{1-m^{*2}}{1-m_-^{*2}}}\,[1+o(1)]\times$$

$$\sum_{\substack{m,m'\in\Gamma_N \\ |m-m^*|<\epsilon,\,|m'-m^*_-|<\epsilon}} \mathrm{e}^{\beta N[f(m)-f(m^*)]-\beta N[f(m')-f(m^*_-)]}.$$

4. To evaluate the sum we write the Taylor expansions

$$f(m) - f(m^*) = \tfrac{1}{2}(m - m^*)^2 f''(m^*) + O((m - m^*)^3),$$
$$f(m') - f(m^*_-) = \tfrac{1}{2}(m' - m^*_-)^2 f''(m^*_-) + O((m' - m^*_-)^3),$$

where we use that $f'(m^*) = 0$ and $f'(m^*_-) = 0$. Changing to new variables $u \equiv \sqrt{N}(m - m^*)$ and $u' \equiv \sqrt{N}(m' - m^*_-)$, we see that the sum equals

$$[1 + o(1)]\,\frac{N}{4} \int_{\mathbb{R}} \mathrm{d}u \int_{\mathbb{R}} \mathrm{d}u' \,\exp\left[\tfrac{1}{2}\beta f''(m^*)u^2 - \tfrac{1}{2}\beta f''(m^*_-)u'^2)\right].$$

Since $f''(m^*) < 0$ and $f''(m^*_+) > 0$, the integral converges and equals

$$\frac{2\pi}{\beta\sqrt{[-f''(m^*)]\, f''(m^*_-)}}.$$

Combining, we end up with the desired formula in Theorem 4.1, but with an extra N in the prefactor. $\square$

The result in THEOREM 4.1 fits the classical Arrhenius law with activation energy $\beta[f(m^*) - f(m_-^*)]$ and with amplitude given by the prefactor.

The activation energy coincides with what was found by Kramers for Brownian motion in a double-well potential, alluded to in Lecture 1.

The prefactor is off by a factor N, which is due to the fact that time is discrete and only one spin is flipped per time step. In the continuous-time version, we speed up time by a factor N, after which the match is perfect.

As a corollary we get the exponential law of the metastable crossover time.

THEOREM 4.2 Bovier, Eckhoff, Gayrard, Klein 2001

As $N \to \infty$,

$$\lim_{N\to\infty} \mathbb{P}_{m_N^-}\left(\frac{\tau_{m_N^+}}{\mathbb{E}_{m_N^-}\left[\tau_{m_N^+}\right]} > t\right) = \mathrm{e}^{-t} \qquad \forall\, t \geq 0.$$

LOSS OF MEMORY!

PAPER:

A. Bovier, M. Eckhoff, V. Gayrard, M. Klein,
Metastability in stochastic dynamics of disordered mean-field models,
Probab. Theory Relat. Fields 119 (2001) 99–161.

A. Bovier, M. Eckhoff, V. Gayrard, M. Klein,
Metastability and low lying spectra in reversible Markov chains,
Commun. Math. Phys. 228 (2002) 219–255.

LITERATURE:

Chapter 13 of Bovier and den Hollander 2015, and references therein.