LECTURE 14

Chung-Lu random graphs

§ TARGET

In this lecture we focus on a class of inhomogeneous Erdős-Rényi random graphs, in particular, on what are called Chung-Lu random graphs.

We will show that the results in Lecture 13 can be extended, and that the inhomogeneity introduces interesting new features to the metastable behaviour of Glauber dynamics on such graphs.

In what follows we first define the model and present a theorem comparing the quenched and the annealed version of the model.

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Afterwards we analyse the annealed version of the model, which is a Glauber dynamics on the complete graph with coupling disorder, and compute two terms in the Arrhenius formula.

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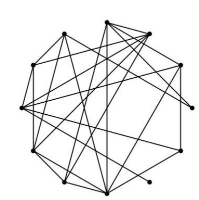
§ CHUNG-LU RANDOM GRAPHS

The Chung-Lu random graph is a generalisation of the Erdős-Rényi random graph that allows for inhomogeneities.

1. Each of N vertices carries an Ising spin that can take the values -1 or $+1$. Let $\mathcal{S}_N = \{-1,+1\}^{[N]}$ denote the set of spin configurations, where $[N] = \{1,\dots,N\}$.

2. Let $\mathcal{P}$ denote a probability measure that is supported on a finite subset of $[0,1]$. Let $J = (J(i))_{i\in[N]}$ be i.i.d. random variables with common law $\mathcal{P}$. Given J, which are referred to as random weights, the Chung-Lu random graph G_N is obtained by independently connecting vertices $i \neq j$ with probability

$$J(i)J(j).$$



3. Consider the following two functions on $\mathcal{S}_N$:

$$H_N(\sigma) = -\frac{1}{N} \sum_{\substack{i,j \in [N] \\ i<j}} 1_{i \sim j}\, \sigma(i)\sigma(j) - h \sum_{i \in [N]} \sigma(i),$$

$$\tilde{H}_N(\sigma) = -\frac{1}{N} \sum_{\substack{i,j \in [N] \\ i<j}} J(i)J(j)\, \sigma(i)\sigma(j) - h \sum_{i \in [N]} \sigma(i).$$

where $i \sim j$ means that i and j are connected in G_N. These are referred to as the quenched and the annealed Hamiltonian, respectively. Note that

$$\tilde{H}_N = \mathbb{E}^N[H_N],$$

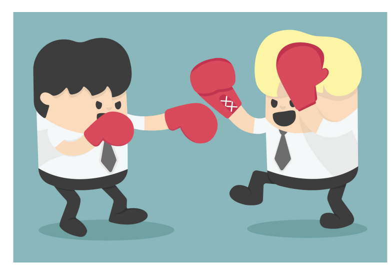
where $\mathbb{E}^N$ denotes expectation over the outcome of G_N for fixed J.

4. Consider Glauber dynamics on $\mathcal{S}_N$ with respect to each Hamiltonian, denoted by

$$(\sigma_N(t))_{t\geq 0},$$
$$(\tilde{\sigma}_N(t))_{t\geq 0}.$$

Each dynamics is reversible w.r.t. the Gibbs measure of its own Hamiltonian at inverse temperature β, given by

$$\mu_N(\sigma) = \frac{1}{Z_N}\,\mathrm{e}^{-\beta H_N(\sigma)},$$
$$\tilde{\mu}_N(\sigma) = \frac{1}{\tilde{Z}_N}\,\mathrm{e}^{-\beta \tilde{H}_N(\sigma)}.$$



5. For $K \in \mathbb{N}$, let

$$\mathcal{M}_N = \{\mathcal{M}_{k,N}\}_{k\in[K]}$$

be a sequence of disjoint metastable sets in $\mathcal{S}_N$. The quenched dynamics $(\sigma_N(t))_{t\geq 0}$ is called ρ_N-metastable with respect to $\mathcal{M}_N$ if

$$K \, \frac{\max_{k\in[K]} \mathbb{P}^N_{\mu_N|\mathcal{M}_{k,N}}\left[\tau_{\mathcal{M}\setminus\mathcal{M}_{k,N}} < \tau_{\mathcal{M}_{k,N}}\right]}{\min_{\mathcal{X}\subset\mathcal{S}_N\setminus\mathcal{M}_N} \mathbb{P}^N_{\mu_N|\mathcal{X}}\left[\tau_{\mathcal{M}_N} < \tau_{\mathcal{X}}\right]} \leq \rho_N,$$

where $\mu_N|\mathcal{X}$ denotes the Gibbs measure μ_N conditioned on the set $\mathcal{X} \subseteq \mathcal{S}_N$.

Recall Lecture 3 for the role of this definition.

6. We need the following assumption of the annealed model.

ASSUMPTION:

There exist an integer $K \geq 2$ and a constant $c_1 > 0$ such that, for $\mathcal{P}^\infty$-a.s. all J, the following holds:

> There exists an $N_0(J) < \infty$ such that for all $N \geq N_0(J)$ there exists a sequence $\mathcal{M}_N$ such that the annealed dynamics $(\tilde{\sigma}_N(t))_{t\geq 0}$ is $\tilde{\rho}_N$-metastable with respect to $\mathcal{M}_N$ with $\tilde{\rho}_N = \mathrm{e}^{-c_1 N}$.

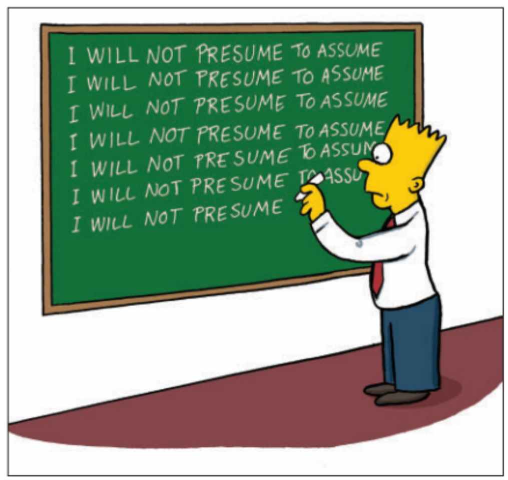
Without loss of generality we can choose the labels of the metastable sets such that

$$k \mapsto \tilde{\mu}_N[\mathcal{M}_{k,N}] \text{ is non-increasing,}$$

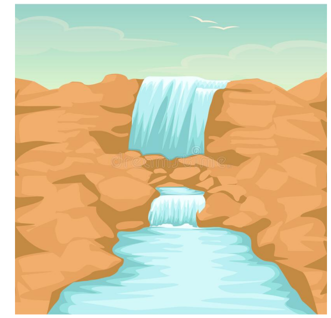
i.e., the larger k the more metastable $\mathcal{M}_{k,N}$. Moreover, we can partition $\mathcal{S}_N = \cup_{k\in[K]}\mathcal{S}_{k,N}$, with $\mathcal{S}_{k,N}$ the valley (= domain of attraction) of $\mathcal{M}_{k,N}$.

7. It can be shown that the quenched dynamics $(\sigma_N(t))_{t\geq 0}$ inherits the properties in the Assumption:

For $\mathcal{P}^{\infty}$-a.s. all J and all $c_2 \in (0, c_1)$ there exists an $N_1(c_2, J) < \infty$ such that for all $N \geq N_1(c_2, J)$ the quenched dynamics $(\sigma_N(t))_{t\geq 0}$ is ρ'_N-metastable with respect to $\mathcal{M}_N$ with $\rho_N = \mathrm{e}^{-c_2 N}$.



§ KEY THEOREM



For $k \in [K] \setminus \{1\}$, put

$$\mathcal{A}_N^k = \mathcal{M}_{k,N}, \qquad \mathcal{B}_N^k = \bigcup_{\ell=1}^{k-1} \mathcal{M}_{\ell,N}.$$

Suppose that there exists a sequence $(\delta_N)_{N\in\mathbb{N}}$ in $[0,1)$ such that $\lim_{N\to\infty} \delta_N = 0$ and

$$\tilde{\mu}_N[\mathcal{S}_{\ell,N}] \leq \delta_N \tilde{\mu}_N[\mathcal{S}_{k,N}] \qquad \forall\, \ell > k.$$

The following theorem bounds the ratio of the average crossover times in the quenched dynamics and the annealed dynamics, starting from their last-exit biased distribution on $\mathcal{A}_N^k$ for the transition from $\mathcal{A}_N^k$ to $\mathcal{B}_N^k$.

THEOREM 14.1 Comparison of quenched vs. annealed

For any $k \in [K] \setminus \{1\}$, $s > 0$ and $\mathcal{P}^\infty$-a.s. all J,

$$\lim_{N\to\infty} \mathbb{P}^N \left[\mathrm{e}^{-s-\alpha_N}[1+o(1)] \leq \frac{\mathbb{E}_{\nu_{\mathcal{A}_N^k,\mathcal{B}_N^k}}[\tau_{\mathcal{B}_N^k}]}{\widetilde{\mathbb{E}}_{\tilde{\nu}_{\mathcal{A}_N^k,\mathcal{B}_N^k}}[\tau_{\mathcal{B}_N^k}]} \leq \mathrm{e}^{s+2\alpha_N}[1+o(1)] \right] \geq 1 - 4\mathrm{e}^{-s^2/\beta^2},$$

where

$$\alpha_N = \alpha_N(J) = \frac{\beta^2}{2N^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} J(i)J(j)[1 - J(i)J(j)].$$

Note that $\lim_{N\to\infty} \alpha_N(J) = \frac{1}{2}\beta^2[m_1^2 - m_2^2]$, $\mathcal{P}^\infty$-a.s., with m_1 and m_2 the first and second moment of $\mathcal{P}$.

The proof of Theorem 4.1 is technical and uses the tools from potential theory that were explained in Lecture 2. The key steps are:

- A concentration inequality for the average crossover time based on a certain bounded difference estimate.
- A variational characterisation that allows for a verification of the bounded difference estimate under the hypothesis, in combination with a comparison of Dirichlet forms.
- An estimate on the Gibbs-norm of the harmonic function.
- Control of random and deterministic metastable partitions.

§ METASTABILITY FOR THE ANNEALED MODEL

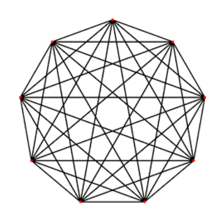
In what follows we analyse the annealed model, which has Hamiltonian

$$\tilde{H}_N(\sigma) = -\frac{1}{N} \sum_{\substack{i,j \in [N] \\ i<j}} J(i)J(j)\,\sigma(i)\sigma(j) - h \sum_{i \in [N]} \sigma(i).$$

This is the Curie-Weiss model on the complete graph, but with coupling disorder. Glauber dynamics $(\tilde{\sigma}(t))_{t \geq 0}$ on $\mathcal{S}_N$ has transition rates

$$\tilde{r}_N(\sigma, \sigma') = \begin{cases} \mathrm{e}^{-\beta[\tilde{H}_N(\sigma') - \tilde{H}_N(\sigma)]_+}, & \text{if } \sigma' \sim \sigma, \\ 0, & \text{otherwise}, \end{cases}$$

where $\sigma' \sim \sigma$ means that σ' differs from σ by a single spin-flip.



▶ Empirical magnetisations

The relevant quantity to monitor is the disorder-weighted magnetisation

$$K_N(\sigma) = \frac{1}{N} \sum_{i \in [N]} J(i)\sigma(i), \qquad \sigma \in \mathcal{S}_N,$$

because the first term in the Hamiltonian equals $\frac{1}{2}[K_N(\sigma)]^2$ up to a constant. Since $\mathcal{P}$ has finite support, we have

$$\mathcal{P} = \sum_{\ell \in [k]} \omega_\ell \delta_{a_\ell}$$

for some $k \in \mathbb{N}$, $(a_\ell)_{\ell \in [k]} \in [0,1]^k$ distinct, and $(\omega_\ell)_{\ell \in [k]} \in (0,1)^k$ such that $\sum_{\ell \in [k]} \omega_\ell = 1$. The following quantities will be crucial for coarse-graining.

Define the level sets

$$A_{\ell,N} = \{i \in [N] \colon\ J(i) = a_\ell\}, \qquad \ell \in [k],$$

and the level magnetisations

$$m_{\ell,N}(\sigma) = \frac{1}{|A_{\ell,N}|} \sum_{i \in A_{\ell,N}} \sigma(i), \qquad \ell \in [k],\ \sigma \in \mathcal{S}_N.$$

Put

$$m_N(\sigma) = \big(m_{\ell,N}(\sigma)\big)_{\ell \in [k]} \in [-1,1]^k, \qquad \sigma \in \mathcal{S}_N,$$

and note that

$$K_N(\sigma) = \frac{1}{N} \sum_{\ell \in [k]} a_\ell \, |A_{\ell,N}| \, m_{\ell,N}(\sigma)$$

depends on σ only through $m_N(\sigma)$. Thus, we may define

$$K_N(m) = \frac{1}{N} \sum_{\ell \in [k]} a_\ell |A_{\ell,N}| m_\ell, \qquad m = (m_\ell)_{\ell \in [k]} \in [-1,1]^k,$$

in particular, $K_N(\sigma) = K_N(m_N(\sigma))$.

▶ Thermodynamic limit

With $\mathcal{P}^{\otimes N}$-probability tending to 1 as $N \to \infty$, the random function K_N converges to a deterministic function K given by

$$K(m) = \sum_{\ell \in [k]} a_\ell \, \omega_\ell \, m_\ell, \qquad m = (m_\ell)_{\ell \in [k]} \in [-1,1]^k.$$

Similarly, the random free energy function F_N converges to a deterministic free energy function $F_{\beta,h}$. These are the k-dimensional analogues of the free energy formulas we saw in Lectures 4 and 13. The stationary points of $F_{\beta,h}$ are given by $\mathbf{m} = (\mathbf{m}_\ell)_{\ell \in [k]}$, where

$$\mathbf{m}_\ell = \tanh(\beta[a_\ell K(\mathbf{m}) + h]), \qquad \ell \in [k].$$

Note that the k-dimensional vector $\mathbf{m}$ is fully determined by the real number $K(\mathbf{m})$. Hence, finding the stationary points of $F_{\beta,h}$ reduces to finding the solutions of

$$K = T_{\beta,h,\mathcal{P}}(K)$$

with

$$T_{\beta,h,\mathcal{P}}(K) = \sum_{\ell\in[k]} a_\ell\,\omega_\ell \tanh(\beta[a_\ell K + h]).$$

- Metastable regime

The metastable regime corresponds to the choice of the parameters $\beta, h, \mathcal{P}$ for which there is more than one solution to the equation $K = T_{\beta,h,\mathcal{P}}(K)$. It turns out that the critical inverse temperature β_c is given by

$$\frac{1}{\beta_c} = \sum_{\ell \in [k]} a_\ell^2 \omega_\ell.$$

Given $\beta \in (\beta_c, \infty)$, the critical magnetic field $h_c(\beta)$ is the minimal value of h for which the system is not metastable. Thus, the metastable regime is

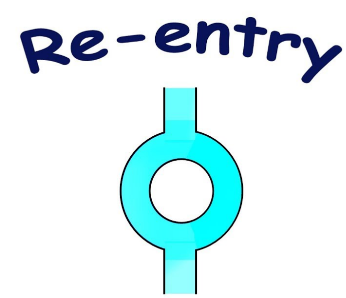
$$\beta \in (\beta_c, \infty), \qquad h \in [0, h_c(\beta)).$$

It can be shown that $\beta \mapsto h_c(\beta)$ is continuous on (β_c, ∞), with

$$\lim_{\beta \downarrow \beta_c} h_c(\beta) = 0, \qquad \lim_{\beta \to \infty} h_c(\beta) = C \in (0, \infty).$$

Interestingly, $\beta \mapsto h_c(\beta)$ is not necessarily monotone, i.e., the metastable crossover may be re-entrant.

It turns out that there exists an $\ell \in [k]$ (depending on $\beta, h, \mathcal{P}$), such that $F_{\beta,h}$ has $2\ell + 1$ stationary points.



▶ Metastable crossover

Let $\mathcal{M}_N$ be the set of minima of F_N. Given $\mathbf{m}_N \in \mathcal{M}_N$, define

$$\mathcal{M}_N(\mathbf{m}_N) \equiv \{m \in \mathcal{M}_N \backslash \mathbf{m}_N \colon\ F_N(m) \leq F_N(\mathbf{m}_N)\}.$$

Fix $\mathbf{m}_N \in \mathcal{M}_N$ as the initial magnetisation.

HYPOTHESES:

(1) $\mathcal{M}_N(\mathbf{m}_N)$ is non-empty.

(2) The Hessian of F_N has only non-zero eigenvalues at $\mathbf{m}_N$ and at all points in the gate $\mathcal{G}(\mathbf{m}_N, \mathcal{M}_N(\mathbf{m}_N))$.

(3) There is a unique $\mathbf{t}_N \in \mathcal{G}(\mathbf{m}_N, \mathcal{M}_N(\mathbf{m}_N))$ that is the saddle point for the crossover.

Let

$$\mathcal{S}_N[\mathbf{m}_N], \qquad \mathcal{S}_N[\mathcal{M}_N(\mathbf{m}_N)],$$

denote the sets of configurations in $\mathcal{S}_N$ for which the level magnetisations are $\mathbf{m}_N$, respectively, are contained in the set $\mathcal{M}_N(\mathbf{m}_N)$. Let $\mathbb{A}_N(\cdot)$ be the $k \times k$ Hessian matrix at the saddle point $\mathbf{t}_N$, and γ_N the unique negative solution of a certain consistency equation at $\mathbf{t}_N$.

For $A \subset \mathcal{S}_N$, write

$$\tau_A = \{t \geq 0\colon\ \sigma_t \in A,\, \sigma_{t-} \notin A\}$$

to denote the first hitting time or return time of A.

THEOREM 14.2 Average crossover time

For every $\mathbf{m_N} \in \mathcal{M}_N$, *uniformly in* $\sigma \in \mathcal{S}_N[\mathbf{m_N}]$ *and with* $\mathcal{P}^{\otimes N}$*-probability tending to* 1 *as* $N \to \infty$,

$$\mathbb{E}_\sigma\left[\tau_{\mathcal{S}_N[\mathcal{M}_N(\mathbf{m_N})]}\right] = [1 + o_N(1)] \times \sqrt{\frac{[-\det(\mathbb{A}_N(\mathbf{t}_N))]}{\det(\mathbb{A}_N(\mathbf{m}_N))}} \left(\frac{\pi}{2\beta(-\gamma_N)}\right) \mathrm{e}^{\beta N[F_N(\mathbf{t}_N) - F_N(\mathbf{m}_N)]}.$$

THEOREM 14.3 Exponential law

For every $\mathbf{m_N} \in \mathcal{M}_N$, *uniformly in* $\sigma \in \mathcal{S}_N[\mathbf{m}_N]$ *and with* $\mathcal{P}^{\otimes N}$*-probability tending to* 1 *as* $N \to \infty$,

$$\mathbb{P}_\sigma\left(\frac{\tau_{\mathcal{S}_N[\mathcal{M}_N(\mathbf{m}_N)]}}{\mathbb{E}_\sigma[\tau_{\mathcal{S}_N[\mathcal{M}_N(\mathbf{m}_N)]}]} > t\right) = [1 + o_N(1)]\,\mathrm{e}^{-t}, \qquad t \geq 0.$$

Note that the average crossover time is a function of the random variable J.

The prefactor converges with $\mathcal{P}^{\otimes N}$-probability tending to 1 as $N \to \infty$ to a deterministic limit, which depends on $\mathcal{P}$ but not on the realisation of J.

The exponent does not converge to a deterministic limit.

In the following theorem we compute the exponent up to order $O(1)$.

We are in a setting where

$$F_N \to F_{\beta,h}, \quad \mathbf{m}_N \to \mathbf{m}, \quad \mathbf{t}_N \to \mathbf{t}, \qquad N \to \infty.$$

THEOREM 14.4 Randomness of the exponent

For every $\mathbf{m_N} \in \mathcal{M}_N$*, in distribution under the law* $\mathcal{P}^{\otimes N}$*,*

$$N[F_N(\mathbf{t}_N) - F_N(\mathbf{m}_N)] = N[F_{\beta,h}(\mathbf{t}) - F_{\beta,h}(\mathbf{m})] + Z\sqrt{N} + O(1),$$

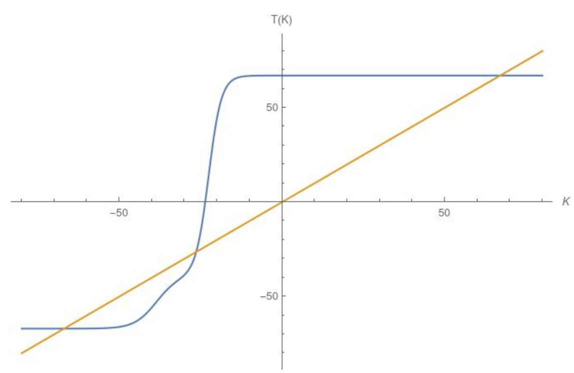
where Z *is a normal random variable with mean* 0 *and variance* $\sigma^2 \in (0,\infty)$*.*

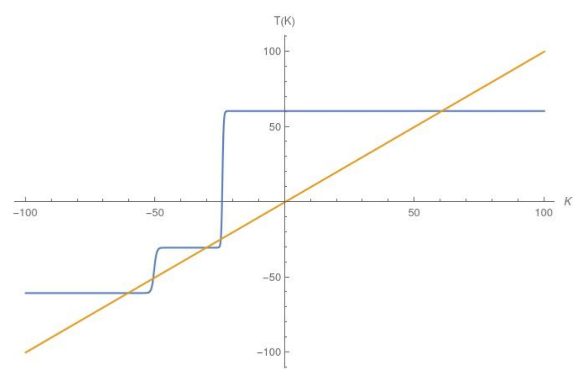
It turns out that σ^2 is a highly complicated function of β, h and $\mathcal{P}$.

The proofs of Theorems 14.2–14.4 rely on a sharp approximation of the Dirichlet form associated with the coarse-grained dynamics. This approximation, together with lumping properties, leads to sharp capacity estimates.

§ NUMERICS: MULTIPLE METASTABLE STATES

The following two figures for $k = 2$ show solutions to the equation $K = T_{\beta,h,\mathcal{P}}(K)$.



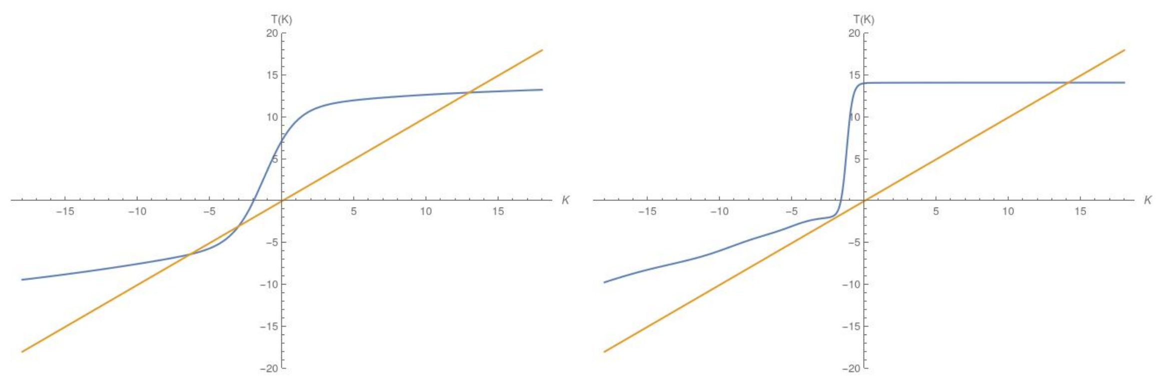


Left: 3 solutions,
parameters: $a_1 = 77$, $a_2 = 45$, $\omega_1 = 0.688$, $h = 1740$, $\beta = 113\,\beta_c$.

Right: 5 solutions,
parameters: $a_1 = 774$, $a_2 = 36.84$, $\omega_1 = 0.59$, $h = 1740$, $\beta = 131\,\beta_c$.

§ NUMERICS: RE-ENTRANT PHASE TRANSITION

The following figures for $k = 4$ show that $\beta \mapsto h_c(\beta)$ is not necessarily increasing.



Pick $a_1 = 12$, $a_2 = 16$, $a_3 = 50.5$, $a_4 = 24.5$,
$\omega_1 = 0.474$, $\omega_2 = 0.22$, $\omega_3 = 0.111$, $h = 100$.

$\beta_1 = 4\,\beta_c = 0.00762336$: 3 solutions, $h < h_c(\beta_1)$.

$\beta_2 = 21\,\beta_c = 0{,}04002264 > \beta_1$: 1 solution, $h > h_c(\beta_2)$.

PAPERS:

(1) A. Bovier, F. den Hollander, S. Marello,
Metastability for Glauber dynamics on the complete graph with coupling disorder,
Commun. Math. Phys. 392 (2022) 307–345.

(2) A. Bovier, F. den Hollander, S. Marello, E. Pulvirenti, M. Slowik,
Metastability for Ising spin-flip dynamics with inhomogeneous coupling disorder,
preprint.