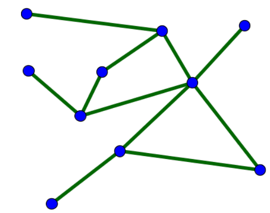
LECTURE 13

Erdős-Rényi random graphs

§ GLAUBER DYNAMICS ON GRAPHS



Let $G = (V, E)$ be a connected graph. Ising spins are attached to the vertices V and interact with each other along the edges E.

1. The energy associated with the configuration $\sigma = (\sigma_i)_{i \in V} \in \Omega = \{-1, +1\}^V$ is given by the Hamiltonian

$$H(\sigma) = -J \sum_{(i,j) \in E} \sigma_i \sigma_j - h \sum_{i \in V} \sigma_i$$

where $J > 0$ is the ferromagnetic interaction strength and $h > 0$ is the external magnetic field.

2. Spins flip according to Glauber dynamics

$$\forall \sigma \in \Omega \; \forall j \in V: \quad \sigma \to \sigma^j \text{ at rate } \mathrm{e}^{-\beta [H(\sigma^j) - H(\sigma)]_+}$$

where σ^j is the configuration obtained from σ by flipping the spin at vertex j, and $\beta > 0$ is the inverse temperature.

3. The Gibbs measure

$$\mu(\sigma) = \frac{1}{\Xi} \mathrm{e}^{-\beta H(\sigma)}, \qquad \sigma \in \Omega,$$

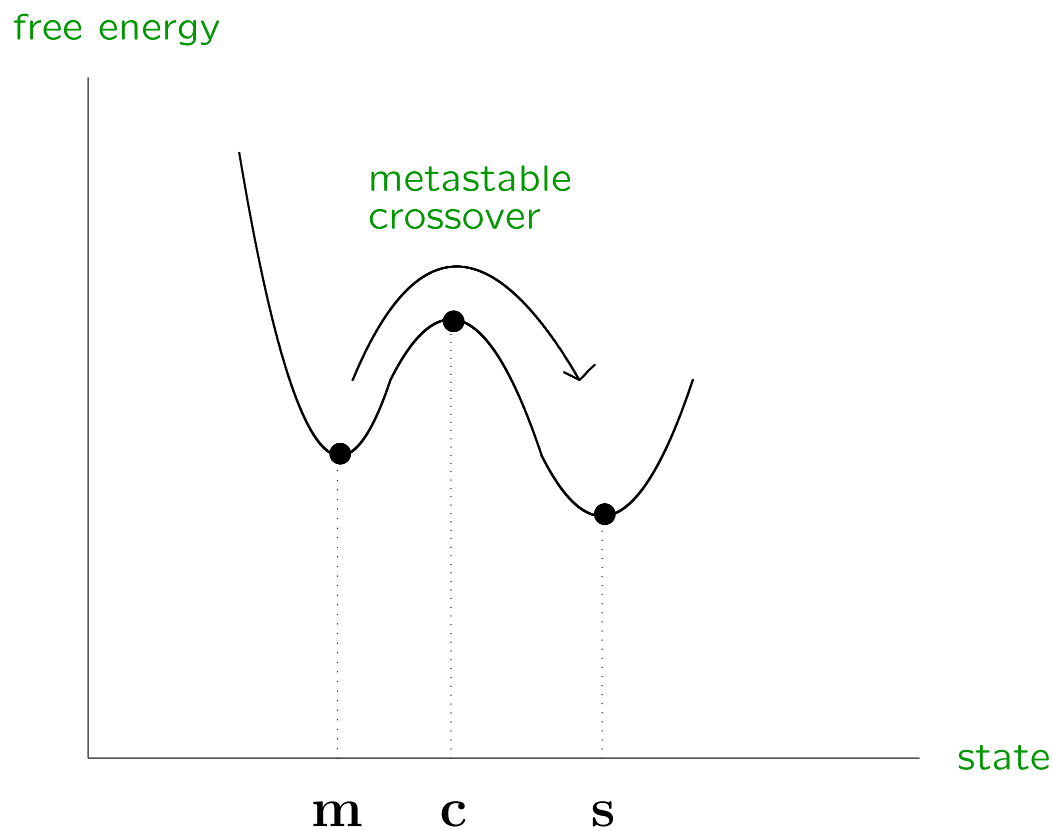
is the reversible equilibrium of this dynamics.

4. Three sets of configurations play a central role:

$$\mathbf{m} = \text{metastable state}$$

$$\mathbf{c} = \text{crossover state}$$

$$\mathbf{s} = \text{stable state}.$$



Caricature of the free energy landscape

– energy and entropy –

§ COMPLETE GRAPH

In Lecture 4 we analysed what happens on the complete graph with N vertices by exploiting a lumping technique.

The ferromagnetic interaction strength was chosen to be $J = N^{-1}$. We found that the empirical magnetisation

$$m_N(t) = \frac{1}{N} \sum_{i \in [N]} \sigma_i(t)$$

performs a continuous-time random walk on the $2N^{-1}$-grid in $[-1, +1]$, in a potential that is given by the finite-volume free energy per vertex

$$f_{\beta,h,N}(m) = -\tfrac{1}{2}m^2 - hm + \beta^{-1} I_N(m)$$

with an entropy term

$$I_N(m) = -\frac{1}{N} \log \binom{N}{\frac{1+m}{2}N}.$$

In the limit $N \to \infty$, the empirical magnetisation performs a Brownian motion on $[-1,+1]$, in a potential that is given by the infinite-volume free energy per vertex

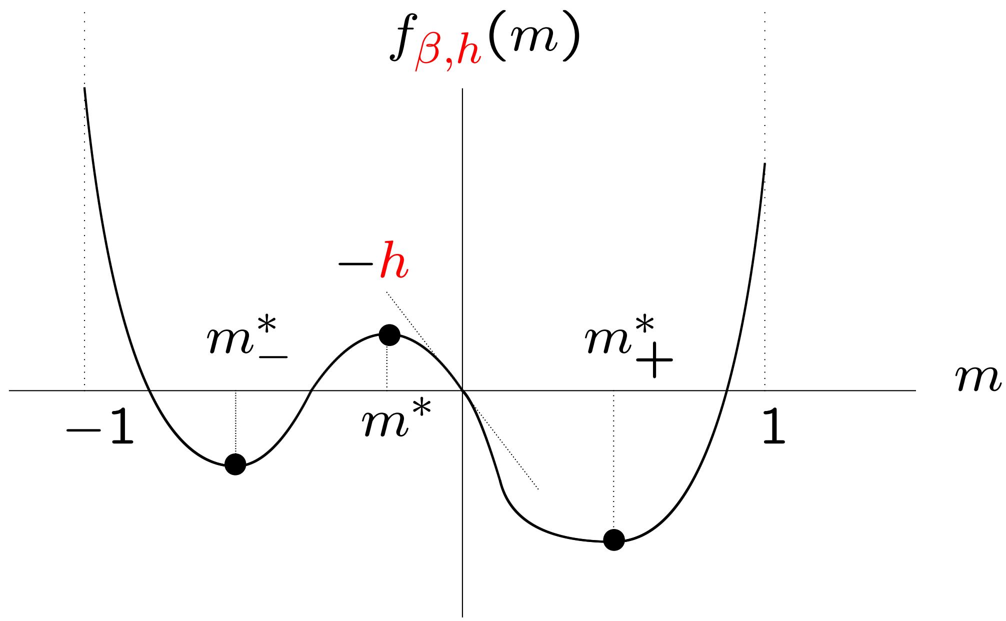
$$f_{\beta,h}(m) = -\tfrac{1}{2}m^2 - hm + \beta^{-1} I(m)$$

with

$$I(m) = \tfrac{1}{2}(1+m)\log(1+m) + \tfrac{1}{2}(1-m)\log(1-m),$$

where a redundant shift by $-\log 2$ is dropped.

The above formulas describe what is called the Curie-Weiss model with Glauber dynamics.



The free energy per vertex $f_{\beta,h}(m)$ at magnetisation m (caricature picture with $\mathbf{m} = m^*_-$, $\mathbf{c} = m^*$, $\mathbf{s} = m^*_+$).

THEOREM 4.1 states that if $\beta > 1$ and $h \in (0, \chi(\beta))$, then

$$\mathbb{E}^{\text{CW}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+}) = K\, \mathrm{e}^{N\Gamma}[1 + o(1)], \qquad N \to \infty,$$

where $\mathbf{m}_N^-, \mathbf{m}_N^+$ are the sets of configurations for which the magnetisation tends to m_-^*, m_+^*,

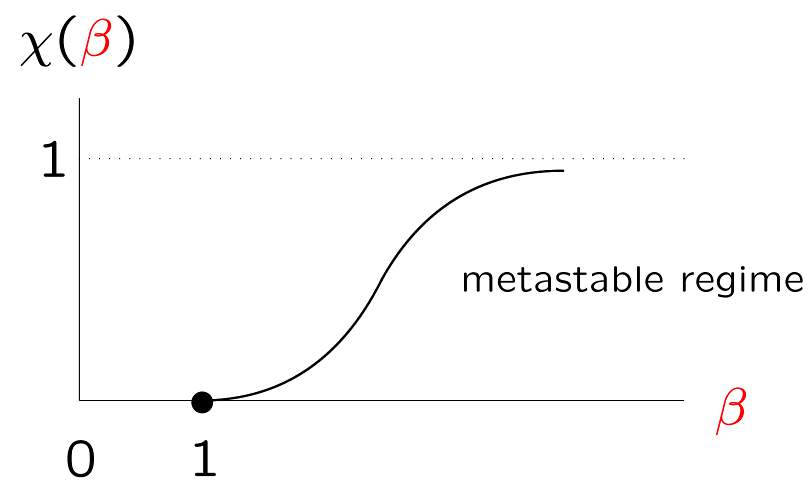
$$\Gamma = \beta\, [f_{\beta,h}(m^*) - f_{\beta,h}(m_-^*)]$$

$$K = \pi\beta^{-1} \sqrt{\frac{1+m^*}{1-m^*}\, \frac{1}{1-m_-^{*2}}\, \frac{1}{[-f''_{\beta,h}(m^*)] f''_{\beta,h}(m_-^*)}}$$

and

$$\chi(\beta) = \sqrt{1 - \tfrac{1}{\beta}} - \tfrac{1}{2\beta} \log\left[\beta\left(1 + \sqrt{1 - \tfrac{1}{\beta}}\right)^2\right].$$

The conditions on β, h guarantee that $f_{\beta,h}$ has a double-well shape and represents the parameter regime for which metastable behaviour occurs.



The expression for the average crossover time in Theorem 4.1 is called the Kramers formula.

§ TARGET

The goal of Lectures 13–15 is to investigate what can be said when the complete graph is replaced by a random graph.

Our target will be to derive an Arrhenius law of the form

$$\mathbb{E}_{\mathbf{m}}[\tau_{\mathbf{s}}] = K\,\mathrm{e}^{N\Gamma}[1+o(1)], \qquad N \to \infty,\ \beta \text{ fixed},$$

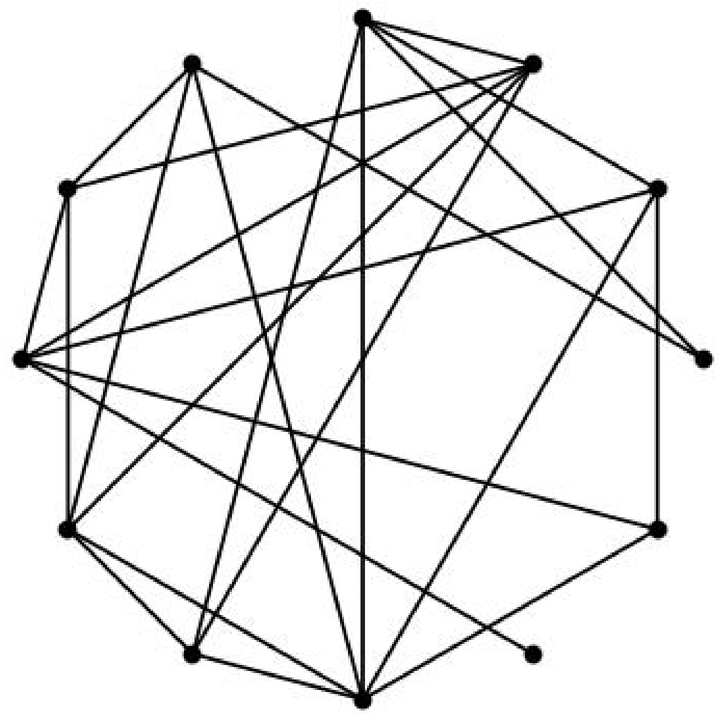
or

$$\mathbb{E}_{\mathbf{m}}[\tau_{\mathbf{s}}] = K\,\mathrm{e}^{\beta\Gamma}[1+o(1)], \qquad \beta \to \infty,\ N \text{ fixed}.$$

However, we will see that in general Γ, K are random and are hard to identify. In fact, in what follows we will mostly have to content ourselves with bounds on these quantities.

In the present lecture we focus on the Erdős-Rényi random graph.

§ ERDŐS-RÉNYI RANDOM GRAPH



Erdős-Rényi random graph: edge percolation

Take the complete graph with N vertices and retain edges with probability $p \in (0, 1)$.

THEOREM 13.1 den Hollander, Jovanovski 2021

On the Erdős-Rényi random graph with N vertices, for $J = 1/pN$, $\beta > 1$ and $h \in (0, \chi(\beta))$,

$$\mathbb{E}^{\mathrm{ER}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+}) = N^{E_N}\, \mathbb{E}^{\mathrm{CW}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+}), \qquad N \to \infty,$$

where E_N is a random exponent that satisfies

$$\lim_{N\to\infty} \mathbb{P}_{\mathrm{ER_N(p)}}\Big(|E_N| \leq \tfrac{11}{6}\tfrac{\beta}{p}(m^* - m_-)\Big) = 1,$$

with $\mathbb{P}_{\mathrm{ER_N(p)}}$ the law of the random graph.

Apart from a polynomial error term, the crossover time is the same on the Erdős-Rényi random graph as on the complete graph, after the change of interaction from $J = 1/N$ to $J = 1/pN$.

The asymptotic estimate of the crossover time is uniform in the starting configuration drawn from the set $\mathbf{m}_N^-$.

Note that J needs to be scaled up by a factor $1/p$ in order to allow for a comparison with the Curie-Weiss model: in the Erdős-Rényi model every spin interacts with $\sim pN$ spins rather than N spins.

On the complete graph the prefactor is constant and computable.

On the Erdős-Rényi random graph it is random and more involved.

ENTROPY

The proof of Theorem 13.1 follows the pathwise approach to metastability mentioned in Lecture 1.

The empirical magnetisation $(h_N(t))_{t \geq 0}$ is monitored on mesoscopic space-time scales. The difficulty is that the lumping technique is no longer available: after projection the Markov property is lost.

The way around this problem is by invoking a coupling argument, namely, to sandwich $(h_N(t))_{t \geq 0}$ between two Curie-Weiss models with a slightly perturbed magnetic field h_N, tending to h as $N \to \infty$. The computations are quite elaborate and are beyond the scope of the present course.

REMARK:

- The pathwise approach works because the Erdős-Rényi random graph is locally homogeneous. This allows us to employ concentration of measure arguments.

- Along the way several approximations have to be made, which is why the prefactor can only be determined up to a polynomial error term.

- The pathwise approach has the advantage that the unit exponential limit law for the crossover time can be derived without much extra effort.

§ REFINEMENT OF THE PREFACTOR

THEOREM 13.2 Bovier, Marello, Pulvirenti 2021

For $\beta > 1$, $h > 0$ *small enough and* $s > 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mathrm{ER}_N(p)} \left(C_1 \mathrm{e}^{-s} \leq \frac{\mathbb{E}^{\mathrm{ER}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+})}{\mathbb{E}^{\mathrm{CW}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+})} \leq C_2 \mathrm{e}^{s} \right) \geq 1 - k_1 \mathrm{e}^{-k_2 s^2},$$

where $k_1, k_2 > 0$ *are absolute constants, and* $C_1 = C_1(p, \beta)$ *and* $C_2 = C_2(p, \beta, h)$.

This theorem shows that the prefactor is a tight random variable, and hence constitutes a considerable sharpening of Theorem 13.1.

The proof of Theorem 13.2 follows the potential-theoretic approach to metastability.

> The local homogeneity of the Erdős-Rényi random graph again plays a crucial role: it turns out that the exact same test functions and test flows that work for the Curie-Weiss model can be used to give sharp upper and lower bounds on the average crossover time.

The better control on the prefactor comes at a price:

- The magnetic field has to be taken small enough.
- The dynamics starts according to the last-exit biased distribution on $\mathbf{m}_N^-$ for the transition from $\mathbf{m}_N^-$ to $\mathbf{m}_N^+$, rather than from an arbitrary configuration in $\mathbf{m}_N^-$.
- The exponential limit law is not automatic.

§ TECHNIQUES

Proofs rely on elaborate techniques:

isoperimetric inequalities
concentration estimates
capacity estimates
coupling techniques
coarse-graining techniques
...

These techniques exploit the fact that in the dense regime the Erdős-Rényi random graph is locally homogeneous.

Homogenisation

§ RANDOM MAGNETIC FIELD

An interesting model is where the randomness sits on the vertices rather than on the edges, namely,

$$H(\sigma) = -\frac{1}{N} \sum_{i,j \in [N]} \sigma_i \sigma_j - \sum_{i \in [N]} h_i \sigma_i,$$

where h_i, $i \in [N]$, are i.i.d. random variables drawn from a common probability distribution ν on $\mathbb{R}$.

Bovier, Eckhoff, Gayrard, Klein 2001 — ν discrete

Bianchi, Bovier, Ioffe 2009 + 2012 — ν continuous

Apart from the error term $1+o(1)$, the prefactor turns out to be constant and to be a somewhat involved function of ν.

Lumping techniques can still be used: in the above papers the magnetisation is monitored on the level sets of the magnetic field. In Lecture 14 we will encounter this technique.

> Our earlier model is much harder because the interaction runs along the set of edges, which has an intricate spatial structure.

TAKE-HOME MESSAGE

Prefactors of average metastable crossover times are delicate objects for random graphs, because they depend in an intricate manner on the underlying geometry.

Very little is known so far and much remains to be done!

Lectures 13–15 offer a few glimpses.

PAPERS:

(1) F. den Hollander, O. Jovanovski,
Glauber dynamics on the Erdős-Rényi random graph,
Progress in Probability 77, Birkhäuser, 2021, pp. 519–589.

(2) A. Bovier, S. Marello, E. Pulvirenti,
Metastability for the dilute Curie-Weiss model with Glauber dynamics,
Electron. J. Probab. 26 (2021) 1–38.