PUTNAM PRACTICE SET 5

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Problem 1. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We consider a function $f : \mathbb{N} \longrightarrow \mathbb{N}_0$ satisfying the following properties:

(a) for any $m, n \in \mathbb{N}$, we have that $f(m+n) - f(m) - f(n) \in \{0, 1\}$
(b) $f(2) = 0$;
(c) $f(3) > 0$; and
(d) $f(9999) = 3333$.

Compute $f(2019)$.

Solution. We get immediately $f(0) = 0$ (from $m = n = 0$). From $m = n = 1$ and $f(0) = 0$, we get $f(1) = 0$. So, using $n = 1$ and m arbitrary, we conclude that $f(m+1) \geq f(m)$ for all m. Since $f(3) > 0$, then using $m = 1$, $n = 2$ along with $f(1) = f(2) = 0$, we conclude that $f(3) = 1$. Using $n = 3$ and m arbitrary, we conclude that $f(m+3) \geq f(m) + 1$ for all $m \geq 0$. In particular, this means that $f(m) \geq \left[\frac{m}{3}\right]$ for all $m \geq 0$. Since $f(9999) = 3333$ we conclude that we must have equality at each step and so, $f(2019) = \left[\frac{2019}{3}\right] = 673$.

Note that if $f(2019) \geq 674$, then inductively we would have gotten that $f(9999) = f(2019 + 3 \cdot 2660) \geq 674 + 2660 > 3333$, contradiction.

Problem 2. Find all real numbers a for which the equation

$$16x^4 - ax^3 + (2a+17)x^2 - ax + 16 = 0$$

has 4 distinct real roots which form a geometric progression.

Solution. We solve equations of the form

$$Ax^4 + Bx^3 + Cx^2 + Bx + A = 0$$

by dividing by x^2 (since $x = 0$ is not a solution because $A \neq 0$) and then defining a new variable $y = x + \frac{1}{x}$. So, in our case, we have

$$16x^2 - ax + (2a+17) - \frac{a}{x} + \frac{16}{x^2} = 0.$$

Letting $y = x + \frac{1}{x}$, we have that $x^2 + \frac{1}{x^2} = y^2 - 2$ and so, our equation above transforms into a quadratic equation:

$$16(y^2 - 2) - ay + (2a+17) = 0 \text{ i.e. } 16y^2 - ay + 2a - 15 = 0.$$

Now, note that since the original quartic equation has 4 distinct real roots, then that means the above quadratic equation has two distinct real roots y_1 and y_2 and then the 4 roots x_i of the quartic equation correspond to solutions of the equations $x + \frac{1}{x} = y_j$. Next, we let x_1 and x_2 be the solutions of the equation $x + \frac{1}{x} = y_1$;

clearly, they satisfy $x_2 = \frac{1}{x_1}$. We also let x_3 and x_4 be the solutions of the equation $x + \frac{1}{x} = y_2$ and so, again $x_4 = \frac{1}{x_3}$. In particular, this means that

$$x_1x_2 = x_3x_4 = 1.$$

Now, since we assumed that the roots x_i form a geometric progression, then they are of the form qr^i for $i = 0, 1, 2, 3$ (for some real numbers q and r). Furthermore, since we also assumed these roots x_i are distinct, then we have that $q \neq 0$ and also, we have that $r \notin \{-1, 0, 1\}$. Now, for the four numbers q, qr, qr^2, qr^3, the only way for the product of two of them to equal the product of the other two numbers would be:

$$q \cdot qr^3 = qr \cdot qr^2$$

since any other combination would force that $r = \pm 1$, contradiction. Therefore, at the expense of relabeling the roots x_i, we have the following:

$$x_1 = q,\ x_2 = qr^3,\ x_3 = qr \text{ and } x_4 = qr^2.$$

In particular, using that $x_1x_2 = x_3x_4 = 1$, we obtain that

$$q^2r^3 = 1; \text{ so } r = \frac{1}{\sqrt[3]{q^2}}.$$

Hence, $x_1 = q$, $x_2 = \frac{1}{q}$, $x_3 = \sqrt[3]{q}$ and $x_4 = \frac{1}{\sqrt[3]{q}}$. Using the fact that x_1 and x_2 are the roots of

$$x^2 - y_1x + 1 = 0,$$

while x_3 and x_4 are the roots of

$$x^2 - y_2x + 1 = 0,$$

we get that

$$q + \frac{1}{q} = y_1 \text{ and } \sqrt[3]{q} + \frac{1}{\sqrt[3]{q}} = y_2.$$

Now, letting $t := \sqrt[3]{q}$ and observing that

$$t^3 + \frac{1}{t^3} = \left(t + \frac{1}{t}\right)^3 - 3\left(t + \frac{1}{t}\right),$$

we conclude that

$$y_1 = y_2^3 - 3y_2.$$

Therefore, in order to have the four roots of the original quartic equation forming a geometric progression, all we need is for the two roots y_1 and y_2 of the quadratic equation

$$16y^2 - ay + 2a - 15 = 0$$

satisfy the relation $y_1 = y_2^3 - 3y_2$. Now, from the Viéte's equations, we have:

$$y_1 + y_2 = \frac{a}{16} \text{ and } y_1y_2 = \frac{2a - 15}{16},$$

which means that

$$y_1y_2 - 2(y_1 + y_2) = -\frac{15}{16} \text{ and so,}$$

$$(y_1 - 2)(y_2 - 2) = \frac{49}{16} = \left(\frac{7}{4}\right)^2.$$

On the other hand, using that $y_1 = y_2^3 - 3y_2$, we have

$$y_1 - 2 = y_2^3 - 3y_2 - 2 = (y_2 - 2)(y_2^2 + 2y_2 + 1) \text{ and so,}$$

$$(y_1 - 2)(y_2 - 2) = (y_2 - 2)^2(y_2 + 1)^2 \text{ and thus, either}$$
$$(y_2 - 2)(y_2 + 1) = \frac{7}{4} \text{ or } (y_2 - 2)(y_2 + 1) = -\frac{7}{4}.$$
Since $y_2 = x_3 + \frac{1}{x_3}$, we have that $|y_2| \geq 2$ (since x_3 is a real number). The only solution in absolute value at least equal to 2 from the above 4 roots of the above 2 quadratic equations is $y_2 = \frac{5}{2}$. This leads to roots x_3 and x_4 being 2 and $\frac{1}{2}$. Furthermore, $y_1 = \frac{125}{8} - \frac{15}{2} = \frac{65}{8}$ and so,
$$a = 16 \cdot (y_1 + y_2) = 16 \cdot \frac{85}{8} = 170.$$

Problem 3. Let $P(x)$ be a monic polynomial of degree 3 with integer coefficients. If one of its roots equals the product of the other two roots, then prove that there exists an integer m such that
$$2P(-1) = m \cdot (P(1) + P(-1) - 2 - 2P(0)).$$

Solution. We have that
$$P(x) = (x - r)(x - s)(x - rs),$$
which means that
$$P(x) := x^3 - (r + s + rs)x^2 + rs(1 + r + s)x - r^2s^2.$$
In particular, $2P(-1) := -2(1 + r)(1 + s)(1 + rs)$, while
$$\begin{aligned} P(1) + P(-1) - 2 - 2P(0) & \\ &= -2(r + s + rs) - 2r^2s^2 - 2 + 2r^2s^2 \\ &= -2(1 + r)(1 + s). \end{aligned}$$
So, the number m from the conclusion is $1+rs$ and thus all we need to show is that $rs \in \mathbb{Z}$. However, we know $P \in \mathbb{Z}[x]$, which means that $r^2s^2 \in \mathbb{Z}$. On the other hand, both $rs(1+r+s)$ and $r+s+rs$ are integers. We let thus $a := r+s+rs \in \mathbb{Z}$ and also, $b := rs(1 + r + s) \in \mathbb{Z}$, which means that
$$b = rs(1 + a - rs)$$
and so, using that $r^2s^2 \in \mathbb{Z}$, we conclude that rs must be a rational number (unless $a = -1$, which would be equivalent with r or s be equal to -1 and then we could take $m = 0$ since $P(-1) = 0$).

Now, if a number z has the property that it is rational and its square is an integer, then it must be an integer itself. The reasoon for this is because otherwise $z = \alpha/\beta$, for some coprime integers α and β and moreover, $\beta > 1$ and then we could take p be a prime number dividing β and not dividing α, which would lead to $z^2 = \alpha^2/\beta^2$ not be an integer since the denominator is divisible by p, while the numerator isn't divisible by p. So, indeed $rs \in \mathbb{Z}$ and thus m is an integer, as claimed.

Problem 4. Let $m, n \in \mathbb{N}$. In a box there are m white balls and n black balls. We extract randomly two balls from the box; if the two balls have different colors, then we put back in the box a white ball, while if the two balls have the same color, then we put back in the box a black ball. We repeat this procedure until there is left in the box only one single ball. What is the probability that this last ball is white?

Solution. We observe that no matter what balls we extract the parity of the number of white balls is always preserved. This means that the last remaining ball is white if and only if m is odd (regardless of how we extracted the balls prior to this last moment).