PUTNAM PRACTICE SET 10

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Problem 1. Prove that for each positive integer n, we have

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < \prod_{i=1}^{n}(2i-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}},$$

where e is base of the natural logarithm.

Solution. Using the fact that $\ln(x)$ is an increasing function, we have that the right rectangles of width 2 represent an overestimate of the area under the graph of $\ln(x)$, while the left rectangles of same width represent an underestimate of that area; so,

$$\int_1^{2n-1} \ln(x)\mathrm{dx} < 2\cdot(\ln(3)+\cdots+\ln(2n-1)) < \int_3^{2n+1}\ln(x)\mathrm{dx}.$$

We have that $\int \ln(x)\mathrm{dx} = x\ln(x) - x + \mathbf{C}$ and therefore

$$((2n-1)\ln(2n-1)-(2n-1))+1 < 2\ln\left(\prod_{i=1}^{2n-1}\right) < ((2n+1)\ln(2n+1)-(2n+1))-(3\ln(3)-3)$$

and then (after exponentiating)

$$(2n-1)^{\frac{2n-1}{2}}e^{-n+1} < \prod_{i=1}^{2n-1} < (2n+1)^{\frac{2n+1}{2}}e^{-n+1}3^{-\frac{3}{2}}.$$

Finally, using in the first inequality that $e>1$, while in the second inequality we use that $3>e$, we derive the desired conclusion.

Problem 2. For any square matrix A with real entries, we can define

$$\sin(A) := \sum_{n=0}^{\infty}\frac{(-1)^n A^{2n+1}}{(2n+1)!},$$

i.e., the above series converges. Determine with proof whether there exists some matrix A with real entries such that

$$\sin(A) = \begin{pmatrix} 1 & 2019 \\ 0 & 1 \end{pmatrix}.$$

Solution. Assume there exists such a matrix A; then there are two cases:

Case 1. A is diagonalizable, i.e., there exists a diagonal matrix D and some invertible matrix B such that $A = B^{-1}DB$.

Then for each positive integer k, we have $A^k = B^{-1}D^kB$. This allows us also very quickly to check that the series defining $\sin(A)$ indeed converges because the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^{2n+1}}{(2n+1)!}$$

converges for any complex number λ. Furthermore, letting $\sin(\lambda)$ be the above expression represented for the series corresponding to λ (if λ is a real number, then we recover the usual sine function, while if λ is a complex number, then $\sin(\lambda) = (e^{i\lambda} - e^{-i\lambda})/2i$). The important thing to note is that in this case we would get also that $\sin(A)$ is a diagonalizable matrix (more precisely, $B\sin(A)B^{-1}$ is a diagonal matrix), which is a contradiction.

Case 2. A is not diagonalizable; so, there exists an invertible matrix B such that $A = B^{-1}JB$, where

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Furthermore, in this case, the unique eigenvalue λ of A must be a real number. Then

$$J^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}.$$

Now, $\sin(A) = B^{-1}\sin(J)B$ and so, A and J must have the same eigenvalues, which in this case is 1. However, the only eigenvalue of $\sin(J)$ is $\sin(\lambda)$, so $\sin(\lambda) = 1$, which means that $\lambda = \frac{\pi}{2} + 2\ell\pi$ for some integer ℓ. Now, we compute the only other nonzero entry (not on the diagonal) in $\sin(J)$; we obtain

$$\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)\cdot\lambda^{2k}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k}}{(2k)!} = \cos(\lambda) = 0,$$

because $\lambda = \frac{\pi}{2} + 2\ell\pi$. In conclusion, there exists no such matrix A.

Problem 3. Let $P \in \mathbb{R}[x]$ with the property that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that there exist polynomials $Q_1, Q_2 \in \mathbb{R}[x]$ such that $P(x) = Q_1(x)^2 + Q_2(x)^2$.

Solution. Our hypothesis yields that each real root of $P(x)$ must have even multiplicity (otherwise there exists a change in sign of $P(x)$ near that root). So, there exist polynomials $Q_0 \in \mathbb{R}[x]$ and $C(x) \in \mathbb{R}[x]$ such that $P(x) = Q_0(x)^2 \cdot C(x)$, where no root of $C(x)$ is real. So, $C(x)$ is a product of quadratic polynomials of the form

$$(x - \alpha - \beta\cdot i)(x - \alpha + \beta\cdot i) = (x-\alpha)^2 + \beta^2$$

for real numbers α and β. Finally, using the identity

$$(S^2 + T^2)(U^2 + V^2) = (SU + TV)^2 + (SV - TU)^2$$

we obtain the desired conclusion.

Problem 4. Let a_n be real numbers so that the following power series expansion holds:

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that for each integer $n \geq 0$, there exists a positive integer m such that $a_{n+1}^2 + a_n^2 = a_m$.

Solution. We have $1 - 2x - x^2 = \big(1 - (1+\sqrt{2})x\big) \cdot \big(1 - (1-\sqrt{2})x\big)$ and so,

$$
\begin{aligned}
&\frac{1}{1-2x-x^2} \\
&\qquad = \frac{1}{(1-(1+\sqrt{2})x)(1-(1-\sqrt{2})x)} \\
&\qquad = \frac{1}{2\sqrt{2}} \cdot \left(\frac{1+\sqrt{2}}{1-(1+\sqrt{2})x} - \frac{1-\sqrt{2}}{1-(1-\sqrt{2})x} \right) \\
&\qquad = \frac{1}{2\sqrt{2}} \cdot \left(\sum_{n=0}^{\infty} (1+\sqrt{2})^{n+1} x^n - \sum_{n=0}^{\infty} (1-\sqrt{2})^{n+1} x^n \right).
\end{aligned}
$$

Thus $a_n = \frac{1}{2\sqrt{2}} \cdot \big((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}\big)$. We compute

$$
\begin{aligned}
&a_{n+1}^2 + a_n^2 \\
&\quad = \frac{1}{8} \cdot \Big((1+\sqrt{2})^{2n+2} + (1-\sqrt{2})^{2n+2} - 2(-1)^{n+1} + (1+\sqrt{2})^{2n} + (1-\sqrt{2})^{2n} - 2(-1)^n \Big) \\
&\quad = \frac{1}{8} \cdot \Big((1+\sqrt{2})^{2n} \cdot (4+2\sqrt{2}) + (1-\sqrt{2})^{2n} \cdot (4-2\sqrt{2}) \Big) \\
&\quad = \frac{1}{2\sqrt{2}} \cdot \Big((1+\sqrt{2})^{2n} \cdot (1+\sqrt{2}) + (1-\sqrt{2})^{2n} \cdot (\sqrt{2}-1) \Big) \\
&\quad = a_{2n+1},
\end{aligned}
$$

as desired.