PUTNAM PRACTICE SET 29: SOLUTIONS

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Problem 1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function with the property that whenever A, B, C and D are the vertices of a square in the cartesian plane, then $f(A) + f(B) + f(C) + f(D) = 0$. Find f.

Solution. We will show that f must be identicaly equal to 0. In order to do this, we take any point P in the plane and show that $f(P) = 0$. For our convenience, we fix our coordinate axes in the plane so that P is the origin $(0, 0)$ (any other choice for the coordinate axes would simply induce a shift in our coordinates but the rest of our argument would be unchanged). Then using the hypothesis regarding f, we have the following:

$$f(1,1) + f(1,-1) + f(-1,-1) + f(-1,1) = 0$$

$$f(1,0) + f(0,1) + f(-1,0) + f(0,-1) = 0$$

$$f(1,1) + f(1,0) + f(0,0) + f(0,1) = 0$$

$$f(1,-1) + f(1,0) + f(0,0) + f(0,-1) = 0$$

$$f(-1,-1) + f(-1,0) + f(0,0) + f(0,-1) = 0$$

$$f(-1,1) + f(-1,0) + f(0,0) + f(0,1) = 0$$

Then adding the last four equations, subtracting the first equation and also subtract twice the second equation yields $f(0, 0) = 0$, as desired.

Problem 2. Functions f, g and h are differentiable on the interval $I = (-1/10, 1/10)$ and on this interval, they satisfy the following relations:

$$f' = 2f^2gh + \frac{1}{gh} \text{ and } f(0) = 1$$

$$g' = fg^2h + \frac{4}{fh} \text{ and } g(0) = 1$$

$$h' = 3fgh^2 + \frac{1}{fg} \text{ and } h(0) = 1.$$

Find an explicit formula for $f(x)$ on the interval I.

Solution. Multiplying the first equation by gh, the second equation by fh and the third equation by fg and then adding together the three new equations, we get

$$(fgh)' = 6(fgh)^2 + 6.$$

We solve for the new function $j := fgh$ which satisfies the differential equation $j' = 6j^2 + 6$ with initial data $j(0) = 1$ and obtain

$$\arctan(j(t)) = 6t + \frac{\pi}{4},$$

which means that $j(t) = \tan(6t + \pi/4)$. Substituting this information in the first equation after we divide that first equation by f, we get:

$$\frac{f'}{f} = 2fgh + \frac{1}{fgh} = 2\tan(6t + \pi/4) + \cot(6t + \pi/4).$$

So, integrating this last equation and again using the intial data that $f(0) = 1$, we obtain:

$$\log(f(t)) = -\frac{\ln(\cos(6t + \pi/4))}{3} + \frac{\ln(\sin(6t + \pi/4))}{6} - \frac{\log(2)}{12}$$

and so,

$$f(t) = 2^{-\frac{1}{12}} \cdot \left(\frac{\sin(6t + \pi/4)}{\cos^2(6t + \pi/4)}\right)^{\frac{1}{6}}.$$

Also, note that on the interval $I = (-1/10, 1/10)$, the above function $f(t)$ is well-defined.

Problem 3. Is there a finite abelian group G with the property that the product of the orders of its elements equals 2^{2021}?

Solution. No, as we will prove below. Using the structure theorem for finite abelian groups, we get that G is a direct product of cyclic groups and since we attempt constructing a group G for which the product of all the orders equals a power of 2, we conclude that G is isomorphic to:

$$\prod_{i=1}^{\ell} \left(\mathbb{Z}/2^{a_i}\mathbb{Z}\right)^{b_i},$$

where $1 \le a_1 < a_2 < \cdots < a_\ell$ (for some $\ell \in \mathbb{N}$) and also, $b_i \in \mathbb{N}$ for each $i = 1, \dots, \ell$.

Inside $\mathbb{Z}/2^a\mathbb{Z}$, we have precisely $2^j - 2^{j-1}$ elements of order 2^j for each $j = 1, \dots, a$, and also, we have the identity element which is of order 1. So, in G, for each $k = 1, \dots, a_\ell$, we have 2^{r_k} elements of order at most 2^k, where

$$r_k := \sum_{i=1}^{\ell} b_i \cdot \min\{a_i, k\}.$$

Therefore, in G we have precisely $2^{r_k} - 2^{r_{k-1}}$ elements of order exactly 2^k (where $r_0 := 0$), which means that our hypothesis leads to the following equation:

$$2021 = \sum_{k=1}^{a_\ell} k \cdot \left(2^{r_k} - 2^{r_{k-1}}\right).$$

We rewrite the above formula as

$$2022 = a_\ell \cdot 2^{a_\ell} - \sum_{k=1}^{a_\ell - 1} 2^{r_k}.$$

Since $\{r_k\}$ is a non-decreasing sequence, it means that 2^{r_1} divides the right hand side of our last equation and so, it must also divide 2022, which means that $r_1 = 1$. Because

$$r_1 = \sum_{i=1}^{\ell} b_i,$$

we conclude that $\ell = 1$ and $b_1 = 1$. So, G is isomorphic to $\mathbb{Z}/2^{a_1}\mathbb{Z}$ and also, using the above formulas, we have:

$$r_k = k \text{ for } k = 2, \ldots, a_\ell.$$

Therefore, our equation transforms into:

$$2022 = a_1 \cdot 2^{a_1} - \sum_{k=1}^{a_1-1} 2^k = a_1 2^{a_1} - 2^{a_1} + 2$$

and so,

$$2020 = (a_1 - 1) \cdot 2^{a_1},$$

which has no solution $a_1 \in \mathbb{N}$ because we would need $a_1 \leq 2$ since 2020 is not divisible by 8.

Problem 4. Let S be a set of rational numbers such that

- $0 \in S$;
- if $x \in S$, then $1 + x \in S$ and also $x - 1 \in S$; and
- if $x \in S \setminus \{0, 1\}$, then $\frac{1}{x(x-1)} \in S$.

Must S contain all rational numbers?

Solution. No, as we will show through the following construction which provides a set S satisfying the above three properties without consisting of the entire $\mathbb{Q}$. So, we let

$$S = \mathbb{Q} \setminus \left\{ n + \frac{2}{5} : n \in \mathbb{Z} \right\}.$$

We check that S meets the above three conditions and this will conclude our proof.

Clearly, $0 \in S$; also, since the set

$$\left\{ n + \frac{2}{5} : n \in \mathbb{Z} \right\}$$

is invariant under subtracting or adding integers, then also the second condition above is met by our set S. As for the third condition, we simply need to ensure that no rational of the form $\frac{1}{x(x-1)}$ for $x \in S$ would be equal to $n + \frac{2}{5}$ for some $n \in \mathbb{Z}$. So, letting $x = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ (and $b > 0$) and $\gcd(a, b) = 1$, we arrive at the equation

$$\frac{b^2}{a^2 - ab} = n + \frac{2}{5}$$

and since $\gcd(b, a - b) = \gcd(b, a) = 1$, we must have that

$$a^2 - ab = \pm 5.$$

Using that $b > 0$, we are left with the only possibilities being

$$(a, b) \in \{(5, 4), (5, 6), (1, 6), (-1, 4)\}.$$

However, each one of these four possibilities does not lead to a numerator for $\frac{b^2}{a^2-ab}$ of the form $5n \pm 2$ (which can be checked either directly or by noting that no *square* of an integer is either 2 or -2 modulo 5). This concludes our proof.